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ON THE INTEGRAL EQUATION OF RENEWAL THEORY

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1. **Introduction.** In this paper we consider the behavior of the solutions of the integral equation

$$u(t) = g(t) + \int_0^t u(t-x)f(x)\,dx, \tag{1.1}$$

where $f(t)$ and $g(t)$ are given non-negative functions.[1] This equation appears, under different forms, in population theory, the theory of industrial replacement and in the general theory of self-renewing aggregates, and a great number of papers have been written on the subject.[2] Unfortunately most of this literature is of a heuristic nature so that the precise conditions for the validity of different methods or statements are seldom known. This literature is, moreover, abundant in controversies and different conjectures which are sometimes supported or disproved by unnecessarily complicated examples. All this renders an orientation exceedingly difficult, and it may therefore be of interest to give a rigorous presentation of the theory. It will be seen that some of the previously announced results need modifications to become correct.

The existence of a solution $u(t)$ of (1.1) could be deduced directly from a well-known result of Paley and Wiener [21] on general integral equations of form (1.1).[3] However, the case of non-negative functions $f(t)$ and $g(t)$, with which we are here concerned, is much too simple to justify the deep methods used by Paley and Wiener in the general case. Under the present conditions, the existence of a solution can be proved in a simple way using properties of completely monotone functions, and this method has also the distinct advantage of showing some properties of the solutions, which otherwise would have to be proved separately. It will be seen in section 3 that the existence proof becomes most natural if equation (1.1) is slightly generalized. Introducing the summatory functions

$$U(t) = \int_0^t u(x)\,dx, \qquad F(t) = \int_0^t f(x)\,dx, \qquad G(t) = \int_0^t g(x)\,dx, \tag{1.2}$$

[1] For the interpretation of the equation cf. section 2.

[2] Lotka's paper [8] contains a bibliography of 74 papers on our subject published before 1939. Yet it is stated that even this list "is not the result of an exhaustive search." At the end of the present paper the reader will find a list of 16 papers on (1.1) which have appeared during the two years since the publication of Lotka's paper.

[3] This has been remarked also by Hadwiger [3].

equation (1.1) can be rewritten in the form

$$U(t) = G(t) + \int_0^t U(t-x)\,dF(x). \tag{1.3}$$

However, (1.3) has a meaning even if $F(t)$ and $G(t)$ are not integrals, provided $F(t)$ is of bounded total variation and the integral is interpreted as a Stieltjes integral. Now for many practical applications (and even for numerical calculations) this generalized form of the integral equation seems to be the most appropriate one and, as a matter of fact, it has sometimes been used in a more or less hidden form (e.g., if all individuals of the parent population are of the same age). Our existence theorem refers to this generalized equation.

We then turn to one of the main problems of the theory, namely the asymptotic behavior of $u(t)$ as $t \to \infty$. It is generally supposed that the solution $u(t)$ "in general" either behaves like an exponential function, or that it approaches in an oscillating manner a finite limit q; the latter case should arise if $\int_0^\infty f(t)\,dt = 1$, thus in particular in the cases of a stable population and of industrial replacement. However, special examples have been constructed to show that this is not always so.[4] In order to simplify the problem and to get more general conditions, we shall first (section 4) consider only the question of convergence in mean, that is to say, we shall study the asymptotic behavior not of $u(t)$ itself but of the mean value $u^*(t) = \frac{1}{t}\int_0^t u(x)\,dx$. The question can be solved completely using only the simplest Tauberian theorems for Laplace integrals. Of course, if $u(t) \to q$ then also $u^*(t) \to q$, but not conversely. The investigation of the precise asymptotic behavior of $u(t)$ is more delicate and requires more refined tools (section 5).

Most of section 6 is devoted to a study of Lotka's well-known method of expanding $u(t)$ into a series of oscillatory components, and it is hoped that this study will help clarify the true nature of this expansion. It will be seen that Lotka's method can be justified (with some necessary modifications) even in some cases for which it was not intended, e.g., if the characteristic equation has multiple or negative real roots, or if it has only a finite number of roots. On the other hand limitations of the method will also become apparent: thus it can occur in special cases that a formal application of the method will lead to a function $u(t)$ which apparently solves the given equation, whereas in reality it is the solution of quite a different equation.

Of course, most of the difficulties mentioned above arise only when the function $f(t)$ has an infinite tail. However, it is known that even computational considerations sometimes require the use of such curves, and, as matter of fact,

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[4] Cf. Hadwiger [2] and also Hadwiger, "Zur Berechnung der Erneuerungsfunktion nach einer Formel von V. A. Kostitzin," *Mitt. Verein. schweizerischer Versich.-Math.*, Vol. 34 (1937), pp. 37–43.

exponential and Pearsonian curves have been used most frequently in connection with (1.1). It will be seen that even in these special cases customary methods may lead to incorrect results. Besides, our considerations show how much the solution $u(t)$ is influenced by the values of $f(t)$ for $t \to \infty$, and, accordingly, that extreme caution is needed in practice. The last section contains some simple remarks on the practical computation of the solution.

2. **Generalities on equations (1.1) and (1.3).** This section contains a few remarks on the meaning of our integral equation and on an alternative form under which it is encountered in the literature. A reader interested only in the abstract theory may pass immediately to section 3.

Equation (1.1) can be interpreted in various ways; the most important among them are the following two:

(i) In the theory of industrial replacement (as outlined in particular by Lotka), it is assumed that each individual dropping out is immediately replaced by a new member of zero age. $f(t)$ denotes the density of the probability at the moment of installment that an individual will drop out at age t. The function $g(t)$ is defined by

$$g(t) = \int_0^t \eta(x) f(t-x)\, dx, \tag{2.1}$$

where $\eta(x)$ represents the age distribution of the population at the moment $t = 0$ (so that the number of individuals of an age between x and $x + \delta x$ is $\eta(x)\delta x + o(\delta x)$). Obviously $g(t)$ then represents the rate of dropping out at time t of individuals belonging to the parent population. Finally, $u(t)$ denotes the rate of dropping out at time t of individuals of the total population. Now each individual dropping out at time t belongs either to the parent population, or it came to the population by the process of replacement at some moment $t - x$ $(0 < x < t)$, and hence $u(t)$ satisfies (1.1). It is worthwhile to note that in this case

$$\int_0^\infty f(t)\, dt = 1, \tag{2.2}$$

since $f(t)$ represents a density of probability.

(ii) In population theory $u(t)$ measures the rate of female births at time $t > 0$. The function $f(t)$ now represents the reproduction rate of females at age t (that is to say, the average number of female descendants born during $(t, t + \delta t)$ from a female of age t is $f(t)\delta t + o(\delta t)$). If $\eta(x)$ again stands for the age distribution of the parent population at $t = 0$, the function $g(t)$ of (2.1) will obviously measure the rate of production of females at time t by members of the parent population. Thus we are again led to (1.1), with the difference, however, that this time either of the inequalities

$$\int_0^\infty f(t)\, dt \lessgtr 1 \tag{2.3}$$

may occur; the value of this integral shows the tendency of increase or decrease in the total population.

Theoretically speaking, $f(t)$ and $g(t)$ are two arbitrary non-negative functions. It is true that $g(t)$ is connected with $f(t)$ by (2.1); but, since the age distribution $\eta(x)$ is arbitrary, $g(t)$ can also be considered as an arbitrarily prescribed function.

It is hardly necessary to interpret the more general equation (1.3) in detail: it is the straightforward generalization of (1.1) to the case where the increase or decrease of the population is not necessarily a continuous process. This form of the equation is frequently better adapted to practical needs. Indeed, the functions $f(t)$ and $g(t)$ are usually determined from observations, so that only their mean values over some time units (years) are known. In such cases it is sometimes simpler to treat $f(t)$ and $g(t)$ as discontinuous functions, using equation (1.3) instead of (1.1). For some advantages of such a procedure see section 7. It may also be mentioned that the most frequently (if not the only) special case of (1.1) studied is that where $g(t) = f(t)$. Now it is apparent from (2.1) that this means that all members of the parent population are of zero age: in this case, however, there is no continuous age-distribution $\eta(x)$. Instead we have to use a discontinuous function $\eta(x)$ and write (2.1) in the form of a Stieltjes integral. Thus discontinuous functions and Stieltjes integrals present themselves automatically, though in a somewhat disguised form, even in the simplest cases.

At this point a remark may be inserted which will prove useful for a better understanding later on (section 6). In the current literature we are frequently confronted not with (1.1) but with

$$u(t) = \int_0^\infty u(t-x)f(x)\,dx, \tag{2.4}$$

together with the explanation that it is asked to find a solution of (2.4) which reduces, for $t < 0$, to a prescribed function $h(t)$. Now such a function, as is known, exists only under very exceptional conditions, and (2.4) is by no means equivalent to (1.1). The current argument can be boiled down to the following. Suppose first that the function $g(t)$ of (1.1) is given in the special form

$$g(t) = \int_t^\infty h(t-x)f(x)\,dx, \tag{2.5}$$

where $h(x)$ is a non-negative function defined for $x < 0$. Since the solution $u(t)$ of (1.1) has a meaning only for $t > 0$, we are free to *define* that $u(-t) = h(-t)$ for $t > 0$. This arbitrary definition, then, formally reduces (1.1) to (2.4). It should be noted, however, that this function $u(t)$ does not, in general, satisfy (2.4) for $t < 0$, for $h(t)$ was prescribed arbitrarily. Thus we are not, after all, concerned with (2.4) but with (1.1), which form of the equation is, by the way, the more general one for our purposes. If there really existed a solution of (2.4) which reduced to $h(t)$ for $t < 0$, we could of course define $g(t)$ by (2.5) and transform (2.4) into (1.1) by splitting the interval $(0, \infty)$ into the subintervals

$(0, t)$ and (t, ∞). However, as was already mentioned, a solution of the required kind does not exist in general. It will also be seen (section 6) that the true nature of the different methods and the limits of their applicability can be understood only when the considerations are based on the proper equation (1.1) and not on (2.4).

3. Existence of solutions.

THEOREM 1. *Let $F(t)$ and $G(t)$ be two finite non-decreasing functions which are continuous to the right*[5]. *Suppose that*

$$F(0) = G(0) = 0, \tag{3.1}$$

and that the Laplace integrals[6]

$$\varphi(s) = \int_0^\infty e^{-st}\, dF(t), \qquad \gamma(s) = \int_0^\infty e^{-st}\, dG(t) \tag{3.2}$$

converge at least for $s > \sigma \geq 0$[7]. *In case that $\lim_{s \to \sigma+0} \varphi(s) > 1$, let $\sigma' > \sigma$ be the root*[8] *of the characteristic equation $\varphi(s) = 1$; in case $\lim_{s \to \sigma+0} \varphi(s) \leq 1$, put $\sigma' = \sigma$.*

Under these conditions there exists for $t > 0$ one and only one finite non-decreasing function $U(t)$ satisfying (1.3). *With this function the Laplace integral*

$$\omega(s) = \int_0^\infty e^{-st}\, dU(t) \tag{3.3}$$

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[5] It is needless to emphasize that this restriction is imposed only to avoid trivial ambiguities.

[6] The integrals (3.2) should be interpreted as Lebesgue-Stieltjes integrals over *open* intervals; thus

$$\varphi(s) = \lim_{\epsilon \to +0} \int_\epsilon^\infty e^{-st}\, dF(t),$$

which implies that $\varphi(s) \to 0$ as $s \to \infty$. Alternatively it can be supposed that $F(t)$ and $G(t)$ have no discontinuities at $t = 0$. Continuity of $F(t)$ at $t = 0$ means that there is no reproduction at zero age. This assumption is most natural for our problem, but is by no means necessary. In order to investigate the case where $F(t)$ has a saltus $c > 0$ at $t = 0$, one should take the integrals (3.2) over the closed set $[0, \infty]$, so that

$$\varphi(s) = c + \lim_{\epsilon \to +0} \int_\epsilon^\infty e^{-st}\, dF(t).$$

It is readily seen that Theorem 1 and its proof remain valid if $0 < c < 1$. However, if $c > 1$, then (1.3) plainly has no solution $U(t)$. The continuity of $G(t)$ at $t = 0$ is of no importance and is not used in the sequel.

[7] The condition is formulated in this general way in view of later applications (cf., e.g., the lemma of section 4). In all cases of practical interest $\sigma = 0$.

[8] $\varphi(s)$ is, of course, monotonic for $s > \sigma$ and tends to zero as $s \to \infty$. In order to ensure the existence of a root of $\varphi(s) = 1$, it is sufficient to suppose that the saltus c of $F(t)$ at $t = 0$ is less than 1 (cf. footnote 6).

converges for $s > \sigma'$, *and*

$$\omega(s) = \frac{\gamma(s)}{1 - \varphi(s)}. \tag{3.4}$$

PROOF: A trivial computation shows that for any finite non-decreasing solution $U(t)$ of (1.3) and any $T > 0$ we have

$$\int_0^T e^{-st}\,dU(t) = \int_0^T e^{-st}\,dG(t) + \int_0^T e^{-sx}\,dF(x) \int_0^{T-x} e^{-st}\,dU(t);$$

herein all terms are non-negative and hence by (3.2)

$$\int_0^T e^{-st}\,dU(t) \leq \gamma(s) + \varphi(s) \int_0^T e^{-st}\,dU(t).$$

Now $\varphi(s) < 1$ for $s > \sigma'$, and hence it is seen that the integral (3.3) exists for $s > \sigma'$ and satisfies (3.4). On the other hand it is well-known that the values of $\omega(s)$ for $s > \sigma'$ determine the corresponding function $U(t)$ uniquely, except for an additive constant, at all points of continuity. However, from (1.3) and (3.1) it follows that $U(0) = 0$ and, since by (1.3) $U(t)$ is continuous to the right, the monotone solution $U(t)$ of (1.3), if it exists, is determined uniquely.

To prove the existence of $U(t)$ consider a function $\omega(s)$ defined for $s > \sigma'$ by (3.4). It is clear from (3.2) that $\varphi(s)$ and $\gamma(s)$ are completely monotone functions, that is to say that $\varphi(s)$ and $\gamma(s)$ have, for $s > \sigma$, derivatives of all orders and that $(-1)^n\varphi^{(n)}(s) \geq 0$ and $(-1)^n\gamma^{(n)}(s) \geq 0$. We can therefore differentiate (3.4) any number of times, and it is seen that $\omega^{(n)}(s)$ is continuous for $s > \sigma'$. Now a simple inductive argument shows that $(-1)^n\omega^{(n)}(s)$ is a product of $\{1 - \varphi(s)\}^{-(n+1)}$ by a finite number of completely monotone functions. It follows that $(-1)^n\omega^{(n)}(s) \geq 0$, so that $\omega(s)$ is a completely monotone function, at least for $s > \sigma'$. Hence it follows from a well-known theorem of S. Bernstein and D. V. Widder[9] that there exists a non-decreasing function $U(t)$ such that (3.3) holds for $s > \sigma'$. Moreover, this function can obviously be so defined that $U(0) = 0$ and that it is continuous to the right. Using $U(t)$ let us form a new function

$$V(t) = \int_0^t U(t - x)\,dF(x). \tag{3.5}$$

$V(t)$ is clearly non-negative and non-decreasing. It is readily verified (and, of course, well-known) that

$$\psi(s) \equiv \int_0^\infty e^{-st}\,dV(t) = \omega(s)\varphi(s).$$

It follows, therefore, from (3.4) that $\psi(s) = \omega(s) - \gamma(s)$, and this implies, by the

[9] This theorem has been repeatedly proved by several authors; for a recent proof cf. Feller [19].

uniqueness theorem for Laplace transforms, that $V(t) = U(t) - G(t)$. Combining this result with (3.5) it is seen that $U(t)$ is a solution of (1.3).

THEOREM 2. *Suppose that $f(t)$ and $g(t)$ are measurable, non-negative and bounded in every finite interval $0 \leq t \leq T$. Let the integrals*

$$\varphi(s) = \int_0^\infty e^{-st} f(t)\,dt, \qquad \gamma(s) = \int_0^\infty e^{-st} g(t)\,dt \tag{3.6}$$

converge for $s > \sigma$. Then there exists one and only one non-negative solution $u(t)$ of (1.1) which is bounded in every finite interval[10]. *With this function the integral*

$$\omega(s) = \int_0^\infty e^{-st} u(t)\,dt \tag{3.7}$$

converges at least for $s > \sigma'$, where $\sigma' = \sigma$ if $\lim_{s \to \sigma+0} \varphi(s) \leq 1$, and otherwise $\sigma' > \sigma$ is defined as the root of the characteristic equation $\varphi(s) = 1$. For $s > \sigma'$ equation (3.4) holds.

If $f(t)$ is continuous except, perhaps, at a finite number of points then $u(t) - g(t)$ is continuous.

PROOF: Define $F(t)$ and $G(t)$ by (1.2). Under the present conditions these functions satisfy the conditions of Theorem 1, and hence (1.3) has a non-decreasing solution $U(t)$. Consider, then, an arbitrary interval $0 \leq t \leq T$ and suppose that in this interval $f(t) < M$ and $g(t) < M$. If $0 \leq t < t + h \leq T$ we have by (1.3)

$$\begin{aligned}
0 &\leq \frac{1}{h}\{U(t+h) - U(t)\} \\
&\equiv \frac{1}{h}\{G(t+h) - G(t)\} + \frac{1}{h}\int_t^{t+h} U(t+h-x) f(x)\,dx \\
&\qquad + \frac{1}{h}\int_0^t \{U(t+h-x) - U(t-x)\} f(x)\,dx \\
&\leq M + MU(T) + \frac{M}{h}\int_0^t \{U(t+h-x) - U(t-x)\}\,dx \\
&= M + MU(T) + \frac{M}{h}\int_t^{t+h} U(y)\,dy - \frac{M}{h}\int_0^h U(y)\,dy \\
&< M + 2MU(T).
\end{aligned}$$

Thus $U(t)$ has bounded difference ratios and is therefore an integral. The derivative $U'(t)$ exists for almost all t and $0 \leq U'(t) \leq M$. Accordingly we can differentiate (1.3) formally, and since $U(0) = 0$ it follows that $u(t) = U'(t)$ satisfies (1.1) for almost all t. However, changing $u(t)$ on a set of measure zero does not affect the integral in (1.1), and since $g(t)$ is defined for all t it is seen that

[10] Without the assumptions of positiveness and boundedness this theorem reduces to a special case of a theorem by Paley and Wiener [21]; cf. section 1, p. 243.

$u(t)$ can be defined, in a unique way, so as to satisfy (1.1) and obtain (1.3). Since the solution of (1.3) was uniquely determined it follows that the solution $u(t)$ is also unique. Obviously equations (3.7) and (3.3) define the same function $\omega(s)$, so that (3.4) holds, and (3.7) converges for $s > \sigma'$.

Finally, if $f(t)$ has only a finite number of jumps, the continuity of $u(t) - g(t)$ becomes evident upon writing (1.1) in the form

$$u(t) - g(t) = \int_0^t u(x)f(t-x)\,dx.$$

4. **Asymptotic properties.** In this section we shall be concerned with the asymptotic behavior as $t \to \infty$ not of $u(t)$ itself but of the mean value $u^*(t) = \frac{1}{t}\int_0^t u(\tau)\,d\tau$. If $u(t)$ tends to the (not necessarily finite) limit C, then obviously also $u^*(t) \to C$, whereas the converse is not necessarily true. For the proof of the theorem we shall need the following obvious but useful

LEMMA: *If $u(t) \geq 0$ is a solution of* (1.1) *and if*

$$u_1(t) = e^{kt}u(t), \qquad f_1(t) = e^{kt}f(t), \qquad g_1(t) = e^{kt}g(t), \tag{4.1}$$

then $u_1(t)$ is a solution of

$$u_1(t) = g_1(t) + \int_0^t u_1(t-x)f_1(x)\,dx.$$

THEOREM 3: *Suppose that using the functions defined in Theorem 2 the integrals*

$$\int_0^\infty f(t)\,dt = a, \qquad \int_0^\infty g(t)\,dt = b, \tag{4.2}$$

are finite.

(i) *In order that*

$$u^*(t) = \frac{1}{t}\int_0^t u(\tau)\,d\tau \to C \tag{4.3}$$

as $t \to \infty$, where C is a positive constant, it is necessary and sufficient that $a = 1$, and that the moment,

$$\int_0^\infty t\,f(t)\,dt = m \tag{4.4}$$

be finite. In this case

$$C = \frac{b}{m}. \tag{4.5}$$

(ii) *If $a < 1$ we have*

$$\int_0^\infty u(t)\,dt = \frac{b}{1-a}. \tag{4.6}$$

(iii) *If $a > 1$ let σ' be the positive root of the characteristic equation $\varphi(s) = 1$* (cf. (3.2)) *and put*[11]

$$\int_0^\infty e^{-\sigma' t} t f(t)\, dt = m_1. \tag{4.7}$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-\sigma'\tau} u(\tau)\, d\tau = \frac{b}{m_1}. \tag{4.8}$$

REMARK: The case $a = 1$ corresponds in demography to a population of stationary size. In the theory of industrial replacement only the case $a = 1$ occurs; the moment m is the average lifetime of an individual. The case $a > 1$ corresponds in demography to a population in which the fertility is greater than the mortality. As is seen from (4.8), in this case the mean value of $u(t)$ increases exponentially. It is of special interest to note that in a population with $a < 1$ the integral (4.6) always converges.

PROOF: By (4.2) and (3.7)

$$\lim_{s\to+0} \phi(s) = a, \qquad \lim_{s\to+0} \gamma(s) = b. \tag{4.9}$$

If $a < 1$, it follows from (3.4) that $\lim \omega(s) = b/(1 - a)$ is finite. Since $u(t) \geq 0$ this obviously implies that (4.6) holds. This proves (ii).

If $a = 1$ and m is finite, it is readily seen that

$$\lim_{s\to+0} \frac{1 - \varphi(s)}{s} = m,$$

and hence by (3.4)

$$\lim_{s\to+0} s\omega(s) = \lim_{s\to+0} \gamma(s) \lim_{s\to+0} \frac{s}{1 - \varphi(s)} = \frac{b}{m}.$$

By a well-known Tauberian theorem for Laplace integrals of non-negative functions[12] it follows that $u^*(t) \to \dfrac{b}{m}$. Conversely, if (4.3) holds it is readily seen that[13]

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[11] (4.2) implies the finiteness of m_1.

[12] Cf. e.g. Doetsch [18], p. 208 or 210.

[13] Indeed, if (4.3) holds and if $U(t)$ is defined by (1.2), then there is a $M = M(\epsilon)$ such that $|U(t) - Ct| < M + \epsilon t$. Now

$$\varphi(s) = s \int_0^\infty e^{-st} U(t)\, dt,$$

and hence

$$s\varphi(s) - C = s^2 \int_0^\infty e^{-st} (U(t) - Ct)\, dt,$$

or

$$|s\varphi(s) - C| \leq s^2 \int_0^\infty e^{-st} (M + \epsilon t)\, dt = sM + \epsilon.$$

$$\lim_{s \to +0} s\omega(s) = C,$$

which in turn implies by (3.4) and (4.9) that

$$\lim_{s \to +0} \frac{1 - \varphi(s)}{s} = \frac{b}{C}.$$

This obviously means that the moment (4.4) exists and equals b/C. This proves (i).

Finally case (iii) reduces immediately to (ii) using the above lemma with $k = -\sigma'$. This finishes the proof.

It may be remarked that the finiteness of the integrals (4.2) is by no means necessary for (4.3). This is shown by the following

EXAMPLE: Let

$$f(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} e^{-1/4t}, \qquad g(t) = \frac{1}{\sqrt{\pi t}} e^{-1/4t}.$$

It is readily seen that with these functions $a = 1$, but $b = \infty$. Now[14] $\varphi(s) = e^{-\sqrt{s}}$ and $\gamma(s) = e^{-\sqrt{s}}/\sqrt{s}$, so that

$$\omega(s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}\,(1 - e^{-\sqrt{s}})}.$$

Thus $s\omega(s) \to 1$ as $s \to +0$, and hence $u^*(t) \to 1$. In this particular case it can even be shown that the solution $u(t)$ itself tends to 1 as $t \to \infty$.

In practice, however, the integrals (4.2) will always exist, and accordingly we restrict the consideration to this case.

5. Closer study of asymptotic properties. In this section we shall deal almost exclusively with the most important special case, namely where

$$\int_0^\infty f(t)\, dt = 1. \tag{5.1}$$

The question has been much discussed whether in this case necessarily $u(t) \to C$ as $t \to \infty$, which statement, if true, would be a refinement of (4.3). Hadwiger [2] has constructed a rather complicated example to show that $u(t)$ does not necessarily approach a limit. Now this can also be seen directly and without any computations. Indeed, if $u(t) \to C$ and if (5.1) holds, then obviously

$$\lim_{t \to \infty} \int_0^t u(t - x) f(x)\, dx = C,$$

and hence it follows from (1.1) that $g(t) \to 0$. In order that $u(t) \to C$ it is therefore

[14] The integrals can be evaluated by elementary methods, and are known; cf. e.g. Doetsch [18], p. 25.

necessary that $g(t) \to 0$, and this proves the assertion. In Hadwiger's example $\limsup g(t) = \infty$, which makes his computations unnecessary.

It can be shown in a similar manner that not even the condition $g(t) \to 0$ is sufficient to ensure that $u(t) \to C$. Some restriction as to the total variation of $f(t)$ seems both necessary and natural (conditions on the existence of derivatives are not sufficient). In the following theorem we shall prove the convergence of $u(t)$ under a condition which is, though not strictly necessary, sufficiently wide to cover all cases of any possible practical interest.

THEOREM 4: *Suppose that with the functions $f(t)$ and $g(t)$ of Theorem 2*

$$\int_0^\infty f(t)\,dt = 1, \qquad \int_0^\infty g(t)\,dt = b < \infty. \tag{5.2}$$

Suppose moreover that there exists an integer $n \geqq 2$ such that the moments

$$m_k = \int_0^\infty t^k f(t)\,dt, \qquad k = 1, 2, \cdots, n, \tag{5.3}$$

are finite, and that the functions $f(t)$, $tf(t)$, $t^2f(t)$, $\cdots$, $t^{n-2}f(t)$ are of bounded total variation over $(0, \infty)$. Suppose finally that

$$\lim_{t\to\infty} t^{n-2}g(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} t^{n-2}\int_t^\infty g(x)\,dx = 0. \tag{5.4}$$

Then

$$\lim_{t\to\infty} u(t) = \frac{b}{m_1} \tag{5.5}$$

and

$$\lim_{t\to\infty} t^{n-2}\left\{u(t) - \frac{b}{m_1}\right\} = 0. \tag{5.6}$$

REMARK: As it was shown in section 4, the case where $\int_0^\infty f(t)\,dt > 1$ can readily be reduced to the above theorem by applying the lemma of section 4 with $k = \sigma'$, where σ' is the positive root of $\varphi(s) = 1$: it is only necessary to suppose that $e^{-\sigma' t}f(t)$ is of bounded total variation and that $e^{-\sigma' t}g(t) \to 0$. Obviously all moments of $e^{-\sigma' t}f(t)$ exist, so that the above theorem shows that $u_1(t) = e^{-\sigma' t}u(t)$ tends to the finite limit b'/m_1', where

$$b' = \int_0^\infty e^{-\sigma' t}g(t)\,dt, \qquad m_1' = \int_0^\infty e^{-\sigma' t}tf(t)\,dt.$$

Thus in this case and under the above assumptions $u(t) \sim \frac{b'}{m_1'}e^{\sigma' t}$, so that the renewal function increases exponentially as could be expected. If however

$$\int_0^\infty f(t)\,dt < 1,$$

$u(t)$ will in general *not* show an exponential character. If $f(t)$ is of bounded variation and has a finite moment of second order, and if $g(t) \to 0$, then it can be shown that $u(t) \to 0$. However, the lemma of section 4 can be applied only if the integral defining $\varphi(s)$ converges in some negative s-interval containing a value s' such that $\varphi(s') = 1$, and this is in general not the case.

PROOF: The proof of Theorem 4 will be based on a Tauberian theorem due to Haar[15]. With some specializations and obvious changes this theorem can be formulated as follows.

Suppose that $l(t)$ is, for $t \geq 0$, non-negative and continuous, and that the Laplace integral

$$\lambda(s) = \int_0^\infty e^{-st} l(t)\, dt \tag{5.7}$$

converges for $s > 0$. Consider $\lambda(s)$ as a function of the complex variable $s = x + iy$ and suppose that the following conditions are fulfilled:

(i) For $y \neq 0$ the function $\lambda(s)$ (which is always regular for $x > 0$) has continuous boundary values $\lambda(iy)$ as $x \to +0$, for $x \geq 0$ and $y \neq 0$

$$\lambda(s) = \frac{C}{s} + \psi(s), \tag{5.8}$$

where $\psi(iy)$ has finite derivatives $\psi'(iy), \cdots, \psi^{(r)}(iy)$ and $\psi^{(r)}(iy)$ is bounded in every finite interval;

(ii) $$\int_{-\infty}^{+\infty} e^{ity} \lambda(x + iy)\, dy$$

converges for some fixed $x > 0$ uniformly with respect to $t \geq T > 0$;

(iii) $\lambda(x + iy) \to 0$ as $y \to \pm\infty$, uniformly with respect to $x \geq 0$;

(iv) $\lambda'(iy), \lambda''(iy), \cdots, \lambda^{(r)}(iy)$ tend to zero as $y \to \pm\infty$;

(v) The integrals

$$\int_{-\infty}^{y_1} e^{ity} \lambda^{(r)}(iy)\, dy \quad \text{and} \quad \int_{y_2}^{\infty} e^{ity} \lambda^{(r)}(iy)\, dy$$

(where $y_1 < 0$ and $y_2 > 0$ are fixed) converge uniformly with respect to $t \geq T > 0$.

Under these conditions

$$\lim t^r \{l(t) - C\} = 0. \tag{5.9}$$

Now the hypotheses of this theorem are too restrictive to be applied to the solution $u(t)$ of (1.1). We shall therefore replace (1.1) by the more special equation

$$v(t) = h(t) + \int_0^t v(t - x) f(x)\, dx, \tag{5.10}$$

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[15] Haar [20] or Doetsch [18], p. 269.

where

$$h(t) = \int_0^t f(t-x)f(x)\,dx. \tag{5.11}$$

Plainly Theorem 2 can be applied to (5.10). It is also plain that $h(t)$ is bounded and non-negative and that (by (5.1))

$$\int_0^\infty h(t)\,dt = 1, \tag{5.12}$$

$$\chi(s) \equiv \int_0^\infty e^{-st}h(t)\,dt = \varphi^2(s). \tag{5.13}$$

Accordingly we have by Theorem 2

$$\zeta(s) \equiv \int_0^\infty e^{-st}v(t)\,dt = \frac{\varphi^2(s)}{1-\varphi(s)}. \tag{5.14}$$

We shall first verify that $\zeta(s)$ satisfies the conditions of Haar's theorem with $r = n-2$. For this purpose we write

$$f(t) = f_1(t) - f_2(t), \tag{5.15}$$

where $f_1(t)$ and $f_2(t)$ are non-decreasing and non-negative functions which are, by assumption, bounded:

$$0 \leq f_1(t) < M, \qquad 0 \leq f_2(t) < M. \tag{5.16}$$

(a) We show that $v(t)$ is continuous. Now by Theorem 2 the solution $v(t)$ of (5.10) is certainly continuous if $h(t)$ is continuous; however, that $h(t)$ is continuous follows directly from (5.11) and the fact that the functions

$$\int_0^t f_1(t-x)f(x)\,dx \quad \text{and} \quad \int_0^t f_2(t-x)f(x)\,dx$$

are continuous.

(b) In view of (5.1) the function $\varphi(s)$ exists for $x = \Re(s) \geq 0$. Obviously $|\varphi(x+iy)| < 1$ for $x > 0$. Now

$$\begin{aligned} 1 - \varphi(iy) &= \int_0^\infty (1 - e^{-iyt})f(t)\,dt \\ &= \int_0^\infty (1-\cos yt)f(t)\,dt + i\int_0^\infty \sin yt \cdot f(t)\,dt, \end{aligned}$$

and, since $1 - \cos yt \geq 0$ and $f(t) \geq 0$, the equality $\varphi(iy) = 1$ for $y \neq 0$ would imply that $f(t) = 0$ except on a set of measure zero. It is therefore seen that $\varphi(x+iy) \neq 1$ for all $x > 0$ and for $x = 0$, $y \neq 0$.

It follows furthermore from (5.3) that for $k = 1, \cdots, n$ and $x \geq 0$ the derivatives

$$\varphi^{(k)}(s) = \int_0^\infty (-t)^k e^{-st}f(t)\,dt$$

exist and that

$$\lim_{x \to +0} \varphi^{(k)}(x + iy) = \varphi^{(k)}(iy).$$

Finally, it is readily seen that in the neighborhood of $y = 0$ we have

$$\varphi(iy) = \int_0^\infty e^{-iyt} f(t)\, dt$$

$$= 1 - m_1 iy + \frac{m_2}{2}(iy)^2 - + \cdots + (-1)^{n-1} \frac{m_{n-1}}{(n-1)!}(iy)^{n-1} + O(|y|^n). \tag{5.17}$$

(c) From what was said under (b) it follows by (5.14) that $\zeta(s)$ is regular for $x > 0$, and that $\zeta(s), \zeta'(s), \cdots, \zeta^{(n)}(s)$ approach continuous boundary values as $s = x + iy$ approaches a point of the imaginary axis other than the origin. Now put

$$\psi(s) = \frac{\varphi^2(s)}{1 - \varphi(s)} - \frac{1}{m_1 s}, \tag{5.18}$$

so that by (5.14)

$$\zeta(s) = \frac{1}{m_1 s} + \psi(s). \tag{5.19}$$

For $x > 0$ and $x = 0$, $y \neq 0$ the function $\psi(x + iy)$ is obviously continuous; the derivatives $\psi'(iy), \cdots, \psi^{(n)}(iy)$ exist. To investigate the behavior of $\psi(iy)$ in the neighborhood of $y = 0$ put

$$P(y) = m_1 - \frac{m_2}{2}(iy) + - \cdots - (-1)^{n-1} \frac{m_{n-1}}{(n-1)!}(iy)^{n-2}. \tag{5.20}$$

By (5.17), (5.18) and (5.20)

$$\psi(iy) = \left[\frac{\{1 - iyP(y)\}^2}{P(y)} - \frac{1}{m_1} \right] \frac{1}{iy} + O(|y|^{n-2}). \tag{5.21}$$

Now the expression in brackets represents an analytic function of y which vanishes at $y = 0$. Hence $\psi(iy) = \mathfrak{P}(y) + O(|y|^{n-2})$, where $\mathfrak{P}(y)$ denotes a power series. It follows that the derivatives $\psi'(iy), \cdots, \psi^{(n-2)}(iy)$ exist for all real y (including $y = 0$) and are bounded for sufficiently small $|y|$: since they are continuous functions they are bounded in every finite interval.

(d). Next we show that there exists a constant $A > 0$ such that for sufficiently large $|y|$

$$|\varphi(x + iy)| < \frac{A}{|y|} \tag{5.22}$$

uniformly in $x \geq 0$. By (5.15)

$$\varphi(s) = \int_0^\infty \{\cos yt - i \sin yt\} e^{-xt} \{f_1(t) - f_2(t)\}\, dt. \tag{5.23}$$

Now $f_1(t)$ is non-decreasing and accordingly by the second mean-value theorem we have for any $T > 0$ and y

$$\int_0^T \cos yt \cdot f_1(t)\, dt = f_1(T) \int_\tau^T \cos yt\, dt = f_1(T) \frac{\sin Ty - \sin \tau y}{y},$$

where τ is some value between 0 and T (depending, of course, on y; at points of discontinuity, $f_1(T)$ should be replaced by $\lim_{t \to T-0} f_1(t)$). Hence by (5.16)

$$\left| \int_0^\infty \cos yt \cdot e^{-xt} \cdot f_1(t)\, dt \right| < \frac{2M}{|y|}.$$

Treating the other terms in (5.23) in a like manner, (5.22) follows.

Combining (5.22) with (5.14) it is seen that for sufficiently large $|y|$

$$|\zeta(s)| < \frac{2A^2}{y^2}$$

uniformly in $x \geq 0$. This shows that the assumptions (ii) and (iii) of Haar's theorem are satisfied for $\lambda(s) = \zeta(s)$. In order to prove that also conditions (iv) and (v) are satisfied it suffices to notice that the proof of (5.22) used only the fact that $f(t)$ is of bounded total variation. Now $\varphi^{(k)}(s)$ is the Laplace transform of $(-t)^k f(t)$, and, since $t^k f(t)$ is of bounded total variation for $k \leq n - 2$, it follows that

$$|\varphi^{(k)}(s)| = O(|y|^{-1}), \qquad k = 1, 2, \cdots, n - 2,$$

for sufficiently large $|y|$, uniformly in $x \geq 0$. Differentiating (5.14) k times it is also seen that

$$|\zeta^{(k)}(s)| = O(|y|^{-2}), \qquad k = 1, 2, \cdots, n - 2,$$

as $y \to +\infty$, uniformly with respect to $x \geq 0$.

This enumeration shows that $v(s) = l(t)$ and $\lambda(s) = \zeta(s)$ satisfy all hypotheses of Haar's theorem with $r = n - 2$ and $C = 1/m_1$. Hence

$$\lim_{t \to \infty} t^{k-2} \left\{ v(t) - \frac{1}{m_1} \right\} = 0. \tag{5.24}$$

Returning now to (5.14) we get

$$\omega(s) = \gamma(s) + \gamma(s)\varphi(s) + \gamma(s)\zeta(s),$$

or, by the uniqueness property of Laplace integrals,

$$\begin{aligned} u(t) &= g(t) + \int_0^t g(x) f(t - x)\, dx + \int_0^t g(x) v(t - x)\, dx \\ &= g(t) + u_1(t) + u_2(t) \end{aligned} \tag{5.25}$$

(which relation can also be checked directly using (5.10)). Let us begin with the last term. We have by (5.2)

$$u_2(t) - \frac{b}{m_1} \equiv \int_0^t g(t-x)\left\{v(x) - \frac{1}{m_1}\right\}dx,$$

and hence

$$t^{n-2}\left|u_2(t) - \frac{b}{m_1}\right| \leq 2^{n-2}\int_{t/2}^t g(t-x)x^{n-2}\left|v(x) - \frac{1}{m_1}\right|dx + t^{n-2}\int_{t/2}^t g(y)\left|v(t-y) - \frac{1}{m_1}\right|dy.$$

If t is sufficiently large we have by (5.24) in the first integral $x^{n-2}\left|v(x) - \frac{1}{m_1}\right| < \epsilon$. In the second integral $v(t-y) - \frac{1}{m_1}$ is bounded, and hence by (5.4)

$$\lim_{t\to\infty} t^{n-2}\left|u_2(t) - \frac{b}{m_1}\right| = 0.$$

The same argument applies (even with some simplifications) also to the second term in (5.24); it follows that

$$\lim t^{n-2}u_1(t) = 0,$$

whilst $t^{n-2}g(t) \to 0$ by assumption (5.4). Now the assertion (5.6) of our theorem follows in view of (5.25) if the last three relationships are added. This finishes the proof of Theorem 4.

It seems that the solution $u(t)$ is generally supposed to oscillate around its limit b/m_1 as $t \to \infty$. It goes without saying that such a behavior is a priori more likely than a monotone character. It should, however, be noticed that there is no reason whatsoever to suppose that $u(t)$ *always* oscillates around its limit. Again no computation is necessary to see this, as shown by the following

EXAMPLE: Differentiating (1.1) formally we get

$$u'(t) = g'(t) + g(0)f(t) + \int_0^t u'(t-x)f(x)\,dx,$$

which shows that, if $g(t)$ and $f(t)$ are sufficiently regular, $u'(t)$ satisfies an integral equation of the same type as $u(t)$. Thus if

$$g'(t) + g(0)f(t) \geq 0$$

for all t, we shall have $u'(t) \geq 0$, and $u(t)$ is a monotone function. In particular, if $g'(t) + g(0)f(t) = 0$, then $u'(t) = 0$ and $u(t) = \text{const}$. For example, let $f(t) = g(t) = e^{-t}$. Then $\varphi(s) = \gamma(s) = 1/(s+1)$ and hence $\omega(s) = 1/s$, which is the Laplace transform of $u(t) = 1$. It is also seen directly that $u(t) \equiv 1$ is the solution. We have however the following

THEOREM 5[16]: *If the functions $f(t)$ and $g(t)$ of Theorem 4 vanish identically for $t \geq T > 0$, then the solution $u(t)$ of (1.1) oscillates around its limit b/m as $t \to \infty$.*

[16] Under some slight additional hypotheses and with quite different methods this theorem was proved by Richter [16].

PROOF: For $t \geq T$ equation (1.1) reduces to

$$u(t) = \int_{t-T}^{t} u(t-x)f(x)\,dx,$$

and since $\int_{t-T}^{t} f(x)\,dx = 1$ it follows that the maxima of $u(t)$ in the intervals $nT < t < (n+1)T$ form, for sufficiently large integers n, a non-increasing sequence. Similarly the corresponding minima do not decrease. Since $u(t) \to b/m_1$, by Theorem 4, it follows that the minima do not exceed b/m_1 and the maxima are not smaller than b/m_1.

6. **On Lotka's method.** Probably the most widely used method for treating equation (1.1) in connection with problems of the renewal theory is Lotka's method. As a matter of fact this method consists of two independent parts. The first step aims at obtaining the exact solution of (1.1) in the form of a series of exponential terms (this is achieved by an adaptation of a method which was used by P. Herz and Herglotz for other purposes. The second part of Lotka's theory consists of devices for a convenient approximative computation of the first few terms of the series. While restricting ourselves formally to Lotka's theory, it will be seen that some of the following remarks apply equally to other methods.

Lotka's method rests essentially on the fundamental assumption that the characteristic equation

$$\varphi(s) = 1 \tag{6.1}$$

has infinitely many distinct simple[17] roots $s_0, s_1, \cdots$, and that the solution $u(t)$ of (1.1) can be expanded into a series

$$u(t) = \sum_k A_k e^{s_k t} \tag{6.2}$$

where the A_k are complex constants. The argument usually rests on an assumed completeness-property of the roots. Thus, starting from (2.4) it is required that (6.2) reduces to $h(t)$ for $t < 0$; in other words, that an arbitrarily prescribed function $h(x)$ be, for $x < 0$, respresentable in the form

$$h(x) = \sum_k A_k e^{s_k x} \qquad (x < 0). \tag{6.3}$$

In practice we are, of course, usually not concerned with $h(t)$ but with $g(t)$ (cf. (2.5)), and according to Lotka's theory the coefficients A_k of the solution (6.2) of (1.1) can be computed directly from $g(t)$ in a way similar to the computation of the Fourier coefficients.

Lotka's method is known to lead to correct results in many cases and also to

[17] Hadwiger [3] objected to the assumption that all roots of (6.1) be simple. The modifications which are necessary to cover the case of multiple roots also will be indicated below.

have distinct computational merits. On the other hand it seems to require a safer justification, since its fundamental assumptions are rarely realized. Thus clearly an arbitrary function $h(x)$ cannot be represented in the form (6.3): to see this it suffices to note that (6.1) frequently has only a finite number of roots (cf. also below). It should also be noted that, the series (6.3) having regularity properties as are assumed in Lotka's theory, any function representable in the form (6.3) is necessarily a solution of the integral equation (2.4), whereas the theory requires us to construct a solution $u(t)$ which reduces to an *arbitrarily* prescribed function $h(t)$ for $t < 0$, (which frequently is an empirical function, determined by observations). Nevertheless, it is possible to give sound foundations to Lotka's method so that it can be used (with some essential limitations and modifications) sometimes even in cases for which it originally was not intended. For this purpose it turns out to be necessary that all considerations be based on the more general equation (1.1), instead of (2.4) (cf. also section 2).

Before proceeding it is necessary to make clear *what is really meant by a root of* (6.1). The function $\varphi(s)$ is defined by (3.2), and the integral will in general converge only for s-values situated in the half-plane $\Re(s) > \sigma$. Usually only roots situated in this half-plane are considered[18]. It is also argued that $\varphi(s)$ is, for real s, a monotone function, so that (6.1) has at most one real root: accordingly the terms of (6.2) are called "oscillatory components." However, the function $\varphi(s)$ can usually be defined by analytic continuation even outside the half-plane $\Re(s) > \sigma$, and, if this is done, (6.1) will in general also have roots in the half-plane $\Re(s) < \sigma$. It will be seen in the sequel that these roots play exactly the same role for the solution $u(t)$ as the other ones, and that the applicability of Lotka's method depends on the behavior of $\varphi(s)$ in the entire complex s-plane. It may be of interest to quote an example where (6.1) has infinitely many real and no other roots.

EXAMPLE[19]: Let

$$f(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} e^{-1/4t}, \qquad t > 0; \tag{6.4}$$

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[18] This was stated in particular by Hadwiger [3] and Hadwiger and Ruchti [6]; accordingly the results of the latter paper (obtained by methods quite different from Lotka's) need some modifications.

[19] Cf. the example at the end of section 4. A function closely related to (6.4) plays an important role in two recent papers by Hadwiger [4] and [5]. Hadwiger's conclusion, if it could be justified, would fundamentally change the aspect of the whole theory. The conclusion reached by Hadwiger seems to be that for any biological population the reproduction function should be of the form $u(t) = \Sigma u_n(t)$, where $u_n(t)$ represents the contribution of the nth generation and

$$(*) \qquad u_n(t) = \frac{an}{\sqrt{\pi}\, t^{3/2}} e^{-At + Can - n^2a^2/t}.$$

Here a, A and C are constants. Clearly (*) is a generalization of (6.4). Now his conclusion is based on the arbitrary assumption that $u_n(t)$ should be of the form $u_n(t) = \psi(x, na)$

It is easily seen that $\varphi(s) = e^{-\sqrt{s}}$. The integral (3.2) converges only for $\Re(s) \geq 0$, but $\varphi(s)$ is defined as a two-valued function in the entire s-plane. The roots of (6.1) are obviously $s_k = -4\,k^2\pi^2$, so that all of them are real and simple. If $g(t) = f(t)$, we get by (3.4)

$$\omega(s) = \frac{e^{-\sqrt{s}}}{1 - e^{-\sqrt{s}}} = \sum_1^\infty e^{-n\sqrt{s}}, \qquad s \text{ real}, > 0.$$

Now $e^{-n\sqrt{s}}$ is the Laplace transform of $\frac{n}{2\sqrt{\pi}\,t^{3/2}} e^{-n^2/4t}$, and hence it is readily seen that the solution $u(t)$ can be written in the form

$$u(t) = \frac{1}{2\sqrt{\pi}\,t^{3/2}} \sum_1^\infty n e^{-n^2/4t}; \tag{6.5}$$

of course, this expansion is not of form (6.2) and shows no oscillatory character.

From now on we shall consistently denote by $\varphi(s)$ the function defined by the integral (3.4) and by the usual process of analytic continuation; accordingly we shall take into consideration *all* roots of (6.1). The main limitation of Lotka's theory can then be formulated in the following way: Lotka's method depends only on the function $g(t)$ and on the roots of (6.1). Now two different functions $f(t)$ can lead to characteristic equations having the same roots. Lotka's method would be applicable to both only if the corresponding two integral equations (1.1) had the same solution $u(t)$. This, however, is not necessarily the case. Thus, if Lotka's method is applied, and if all computations are correctly performed, and if the resulting series for $u(t)$ converges uniformly, there is no possibility of telling which equation is really satisfied by the resulting $u(t)$: it can happen that one has unwittingly solved some unknown equation of type (1.1) which, by chance, leads to a characteristic equation having the same roots as the characteristic equation of the integral equation with which one was really concerned. Indeed this happens in the following example which is familiar in connection with our problem. It is illustrative also for other purposes: thus it shows not only limitations of Lotka's method, but also that this method can be modified so as to become applicable in some cases where the characteristic equation has only a finite number of roots.

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where $\psi(x, a)$ is independent of n. To my mind Hadwiger's result shows only the impractibility of this axiom. However, Hadwiger's result is not correct even under his assumption. Indeed, he derives for $\psi(x, a)$ the functional equation

$$\psi(x, a + b) = \int_0^t \psi(x - \xi, a)\psi(\xi, b)\, d\xi, \tag{**}$$

which is well-known from the theory of stochastic processes. Now Hadwiger merely verifies the known result that (*) leads to a solution of (**). However, (**) has infinitely many other solutions (it is possible to write down expressions for their Laplace transforms, although it is difficult to express the solutions themselves explicitly). This, of course, renders Hadwiger's result illusory.

EXAMPLE: *Pearson type III-curves.*[20] Consider the integral equation (1.1) in the following two cases:

$$\text{(I)} \qquad f(t) = g(t) = f_{\text{I}}(t) = \frac{1}{\Gamma(\frac{3}{2})}\, t^{1/2} e^{-t}$$

and

$$\text{(II)} \qquad f(t) = g(t) = f_{\text{II}}(t) = \tfrac{1}{2} t^2 e^{-t}.$$

It is readily seen (and well known) that the corresponding Laplace transforms are

$$\text{(I)} \qquad \varphi_{\text{I}}(s) = \frac{1}{(s+1)^{3/2}}$$

and

$$\text{(II)} \qquad \varphi_{\text{II}}(s) = \frac{1}{(s+1)^3},$$

respectively. Thus in both cases the characteristic equation has the same roots, namely

$$s_1 = 0, \qquad s_{2,3} = -\frac{3}{2} \pm \frac{i}{2}\sqrt{3},$$

of which only the first one lies in the half-plane of convergence of the integral (3.4). Lotka's method is not applicable since there are only three roots. However, in the second case, an expansion of type (6.2) is possible. Indeed, we have by (3.4)

$$\omega_{\text{II}}(s) = \frac{\varphi_{\text{II}}(s)}{1 - \varphi_{\text{II}}(s)} = \frac{1}{s^3 + 3s^2 + 3s}$$

$$= \frac{1}{3s} - \frac{\frac{1}{6} - \frac{i}{2\sqrt{3}}}{s + \frac{3}{2} - \frac{i}{2}\sqrt{3}} - \frac{\frac{1}{6} + \frac{i}{2\sqrt{3}}}{s + \frac{3}{2} + \frac{i}{2}\sqrt{3}};$$

now $1/(s+a)$ is the Laplace transform of e^{-at}, and hence we obtain the solution $u(t)$ in the form

$$u_{\text{II}}(t) = \frac{1}{3} - \left(\frac{1}{6} - \frac{i}{2\sqrt{3}}\right) e^{\frac{1}{2}(-3+i\sqrt{3})t} - \left(\frac{1}{6} + \frac{i}{2\sqrt{3}}\right) e^{\frac{1}{2}(-3-i\sqrt{3})t}$$

$$= \frac{1}{3} - \frac{1}{3} e^{-3t/2} \cos\frac{\sqrt{3}}{2}\, t - \frac{1}{\sqrt{3}} e^{-3t/2} \sin\frac{\sqrt{3}}{2}\, t,$$

[20] General Pearson curves have been investigated recently in connection with (1.1) by Brown [1], Hadwiger and Ruchti [6] and Rhodes [15]. Hadwiger and Ruchti use a method of their own, but they are also led to the study of the characteristic equation (6.1) in a slightly disguised form: their result needs a modification since they arbitrarily drop the roots lying in the halfplane of divergence of the integral $\varphi(s)$.

which is an expansion of type (6.2). In the first of the above examples we get for real positive s

$$\omega_I(s) = \frac{\varphi_I(s)}{1 - \varphi_I(s)} = \sum_{n=1}^{\infty} \frac{1}{(s+1)^{3n/2}},$$

and it is readily seen that this is the Laplace transform of the solution

$$u_I(t) = e^{-t} \sum_{n=1}^{\infty} \frac{1}{\Gamma(3n/2)} t^{3(n-2)/2}.$$

The series is convergent for $t > 0$, but obviously this solution cannot be represented in a form similar to (6.2).

A similar remark applies to the general Pearson-type III curve

$$f(t) = At^{\beta}e^{-\alpha t},$$

where A, α, β are positive constants; the corresponding Laplace transform is

$$\varphi(s) = A\Gamma(\beta+1) \frac{1}{(s+\alpha)^{\beta+1}}.$$

These preparatory remarks enable us to formulate rigorous conditions for the existence of an expansion of type (6.2). The following theorem shows the limits of Lotka's method, but at the same time it also represents an extension of it. In the formulation of the theorem we have considered only the case of absolute convergence of (6.2). This was done to avoid complications lacking any practical significance whatsoever. The conditions can, of course, be relaxed along customary lines.

THEOREM 6: *In order that the solution $u(t)$ of Theorem 2 be representable in form* (6.2), *where the series converges absolutely for $t \geq 0$ and where the s_k denote the roots of the characteristic equation*[21] (6.1), *it is necessary and sufficient that the Laplace transform $\omega(s)$ admit an expansion*

$$\omega(s) \equiv \frac{\gamma(s)}{1 - \varphi(s)} = \sum \frac{A_k}{s - s_k} \tag{6.6}$$

and that $\Sigma|A_k|$ converges absolutely. The coefficients A_k are determined by

$$A_k = -\frac{\gamma'(s_k)}{\varphi(s_k)}. \tag{6.7}$$

In particular, it is necessary that $\omega(s)$ be a one-valued function.[22]

PROOF: All roots s_k of (6.1) satisfy the inequality $\Re(s_k) \leq \sigma'$, where σ' was defined in Theorem 2. It is therefore readily seen that in case $\Sigma|A_k|$ converges, the Laplace transform of (6.2) can be computed for sufficiently large

[21] The number of roots may be finite or infinite. It should also be noted that it is not required that $s_k \to \infty$. If the s_k have a point of accumulation, $\omega(s)$ will have an essential singularity. That this actually can happen can be shown by examples.

[22] This was not so in our example I.

positive s-values by termwise integration so that (6.6) certainly holds for sufficiently large positive s. Now with $\Sigma \mid A_k \mid$ converging, (6.6) defines $\omega(s)$ uniquely for all complex s (with singularities at the points s_k and the points of accumulation of s_k, if any). Since the analytic continuation is unique, it follows that (6.6) holds for all s. The series $\Sigma \mid A_k \mid$ must, of course, converge if (6.2) is to converge absolutely for $t = 0$, and this proves the necessity of our condition. Conversely, if $\omega(s) = \dfrac{\gamma(s)}{1 - \varphi(s)}$ is given by (6.6), and if $\Sigma \mid A_k \mid$ converges, then $\omega(s)$ is the Laplace transform of a function $u(t)$ defined by (6.2). Since the Laplace transform is unique, $u(t)$ is the solution of (1.1) by Theorem 2. The series (6.2) converges absolutely for $t \geq 0$ since $\mid A_k e^{s_k t} \mid \leq \mid A_k \mid e^{\sigma' t}$. Finally (6.7) follows directly from (6.6).

It is interesting to compare (6.7) with formulas (50) and (56) of Lotka's paper [8]. Lotka considers the special case $g(t) = f(t)$; in this case $\gamma(s_k) = \varphi(s_k) = 1$, and (6.7) reduces to $A_k = -\dfrac{1}{\varphi'(s_k)}$. If s_k lies in the domain of convergence of the integral $\varphi(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$, that is, if $\Re(s_k) \geq \sigma$ then

$$\frac{1}{A_k} = \int_0^\infty e^{-s_k t} u f(t)\, dt, \tag{6.8}$$

in accordance with Lotka's result. However, (6.8) becomes meaningless for the roots with $\Re(s_k) < \sigma$, whereas (6.7) is applicable in all cases.

Theorem 6 can easily be generalized to the case where the *characteristic equation has multiple roots*. The expansion (6.6) (which reduces to the customary expansion into partial fractions whenever $\omega(s)$ is meromorphic) is to be replaced by

$$\omega(s) = \sum_k \left\{ \frac{A_k^{(1)}}{s - s_k} + \frac{A_k^{(2)}}{(s - s_k)^2} + \cdots + \frac{A_k^{(m_k)}}{(s - s_k)^{m_k}} \right\}, \tag{6.9}$$

where m_k is the multiplicity of the root s_k. This leads us formally to an expansion

$$u(t) = \sum_k e^{s_k t} \left\{ A_k^{(1)} + A_k^{(2)} \frac{t}{1!} + \cdots + A_k^{(m_k)} \frac{t^{m_k - 1}}{(m_k - 1)!} \right\}, \tag{6.10}$$

which now replaces (6.2). Generalizing Theorem 6 it is easy to formulate some simple conditions under which (6.11) will really represent a solution of (1.1). Other conditions which ensure that (6.9) is the transform of (6.10) are known from the general theory of Laplace transforms; such conditions usually use only function-theoretical properties of (6.9) and are applicable in particular when $\omega(s)$ is meromorphic. We mention in particular a theorem of Churchill [17] which can be used for our purposes.

7. **On the practical computation of the solution.** There are at hand two main methods for the practical computation of the solution of (1.1). One of them

has been developed by Lotka and consists of an approximate computation of a few coefficients in the series (6.2). The other method uses an expansion

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{7.1}$$

where $u_n(t)$ represents the contribution of the nth "generation" and is defined by x

$$u_0(t) = g(t), \qquad u_{n+1}(t) = \int_0^t u_n(t-x)f(x)\,dx. \tag{7.2}$$

Now the Laplace transform of $u_{n+1}(t)$ is $\gamma(s)\varphi^n(s)$, and hence (7.2) corresponds to the expansion

$$\omega(s) = \frac{\gamma(s)}{1-\varphi(s)} = \gamma(s)\sum_{n=0}^{\infty}\varphi^n(s). \tag{7.3}$$

In practice the functions $g(t)$ and $f(t)$ are usually not known exactly. Frequently their values are obtained from some statistical material, so that only their integrals over some time units, e.g. years, are actually known or, in other words, only the values

$$f_n = \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} f(t)\,dt, \qquad g_n = \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} g(t)\,dt, \tag{7.4}$$

are given, where $\delta > 0$ is a given constant. Ordinarily in such cases some theoretical forms (e.g. Pearson curves) are fitted to the empirical data and equation (1.1) is solved with these theoretical functions. Now such a procedure is sometimes not only very troublesome, but also somewhat arbitrary. Consider for example the limit of $u(t)$ as $t \to \infty$; this asymptotic value is the main point of interest of the theory and all practical computations. However, as has been shown above, this limit depends only on the moments of the first two orders of $f(t)$ and $g(t)$, and, unless the fitting is done by the method of moments, the resulting value will depend on the special procedure of fitting. Accordingly it will sometimes happen that it is of advantage to use the empirical material as it is, and this can, at least in principle, always be done.

If only the values (7.4) are used it is natural to consider $f(t)$ and $g(t)$ as step-functions defined by

$$\left.\begin{aligned} f(t) &= f_n, \\ g(t) &= g_n, \end{aligned}\right\} \quad \text{for} \quad n\delta \le t < (n+1)\delta. \tag{7.5}$$

In practice only a finite number among the f_n and g_n will be different from zero: accordingly the Laplace transforms $\gamma(s)$ and $\varphi(s)$ reduce to trigonometrical polynomials, so that the analytic study of $\omega(s) = \dfrac{\gamma(s)}{1-\varphi(s)}$ becomes particularly simple. Lotka's method can be applied directly in this case.

For a convenient computation of (7.1) it is better to return to the more general equation (1.3), instead of (1.1). The summatory functions $F(t)$ and $G(t)$ should not be defined by (1.2) in this case, but simply by

$$F(t) = \sum_{n=0}^{\left[\frac{t}{\delta}\right]} f_n, \qquad G(t) = \sum_{n=0}^{\left[\frac{t}{\delta}\right]} g_n. \tag{7.6}$$

It is readily seen that the solution $U(t)$ of (1.3) can be written in the form $U(t) = \sum_{n=0}^{\infty} U_n(t)$, where

$$U_0(t) = G(t), \qquad U_{n+1}(t) = \int_0^t U_n(t-x)\,dF(x);$$

in our case $U_n(t)$ will again be a step-function with jumps at the points $k\delta$, the corresponding saltus being

$$u_0^{(k)} = g_k, \qquad u_{n+1}^{(k)} = \sum_{r=0}^{k} u_n^{(k-r)} f_r.$$

Thus we arrive at exactly the same result as would have been obtained if the integrals (7.2) had been computed, starting from (7.4), by the ordinary methods for numerical integration of tabulated functions. It is of interest to note that this method of approximate evaluation of the integrals (7.2) leads to the *exact values of the renewal function* of a population where all changes occur in a discontinuous way at the end of time intervals of length δ in such a way that each change equals the mean value of the changes of the given population over the corresponding time interval.

REFERENCES

I. Papers on the integral equation of renewal theory

Note: Lotka's paper [8] contains a list of 74 papers on the subject published before 1939. The following list is to bring Lotka's list up to June 1941; however no claims to completeness are made.

[1] A. W. Brown, A note on the use of a Pearson type III function in renewal theory," *Annals of Math. Stat.* Vol. 11 (1940), pp. 448–453.

[2] H. Hadwiger, "Zur Frage des Beharrungszustandes bei kontinuierlich sich erneuernden Gesamtheiten," *Archiv f. mathem. Wirtschafts- und Sozialforschung*, Vol. 5 (1939), pp. 32–34.

[3] H. Hadwiger, "Über die Integralgleichung der Bevölkerungstheorie," *Mitteilungen Verein. schweizer Versicherungsmathematiker* (*Bull. Assoc. Actuaires suisses*), Vol. 38 (1939), pp. 1–14.

[4] H. Hadwiger, "Eine analytische Reproduktionsfunktion für biologische Gesamtheiten," *Skand. Aktuarietidskrift* (1940), pp. 101–113.

[5] H. Hadwiger, "Natürliche Ausscheidefunktionen für Gesamtheiten und die Lösung der Erneuerungsgleichung," *Mitteilungen Verein. Schweiz. Versich.-Math.*, Vol. 40 (1940), pp. 31–39.

[6] H. HADWIGER and W. RUCHTI, "Über eine spezielle Klasse analytischer Geburtenfunktionen," *Metron*, Vol. 13 (1939), No. 4, pp. 17–26.

[7] A. LINDER, "Die Vermehrungsrate der stabilen Bevölkerung," *Archiv f. mathem. Wirtschafts- und Sozialforschung*, Vol. 4 (1938), pp. 136–156.

[8] A. LOTKA, "A contribution to the theory of self-renewing aggregates, with special reference to industrial replacement," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 1–25.

[9] A. LOTKA, "On an integral equation in population analysis," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 144–161.

[10] A. LOTKA, "Théorie analytique des associations biologiques II," *Actualités Scientifiques No. 780*, Paris, 1939.

[11] A. LOTKA, "The theory of industrial replacement," *Skand. Aktuarietidskrift* (1940), pp. 1–14.

[12] A. LOTKA, "Sur une équation intégrale de l'analyse démographique et industrielle," *Mitt. Verein. Schweiz. Versich.-Math.*, Vol. 40 (1940), pp. 1–16.

[13] H. MÜNZNER, "Die Erneuerung von Gesamtheiten," *Archiv f. math. Wirtschafts- u. Sozialforschung*, Vol. 4 (1938).

[14] G. A. D. PREINREICH, "The theory of industrial replacement," *Skand. Aktuarietidskrift* (1939), pp. 1–19.

[15] E. C. RHODES, "Population mathematics, I, II, III," *Roy. Stat. Soc. Jour.*, Vol. 103 (1940), pp. 61–89, 218–245, 362–387.

[16] H. RICHTER, "Die Konvergenz der Erneuerungsfunktion," *Blätter f. Versicherungsmathematik*, Vol. 5 (1940), pp. 21–35.

[16a] H. HADWIGER, "Über eine Funktionalgleichung der Bevölkerungstheorie und eine spezielle Klasse analytischer Lösungen," *Bl. f. Versicherungsmathematik*, Vol. 5 (1941), pp. 181–188.

[16b] G. A. D. PRIENREICH, "The present status of renewal theory," Waverly Press, Baltimore (1940).

II. Other papers quoted

[17] R. V. CHURCHILL, "The inversion of the Laplace transformation by a direct expansion in series and its application to boundary-value problems," *Math. Zeits.*, Vol. 42 (1937), pp. 567–579.

[18] G. DOETSCH, *Theorie und Anwendung der Laplace Transformation*. J. Springer, Berlin, 1937.

[19] W. FELLER, "Completely monotone functions and sequences," *Duke Math. Jour.*, Vol. 5 (1939), pp. 661–674.

[20] A. HAAR, "Über asymptotische Entwicklungen von Funktionen," *Math. Ann.*, Vol. 96 (1927), pp. 69–107.

[21] R. E. A. C. PALEY and N. WIENER, "Notes on the theory and application of Fourier transforms, VII. On the Volterra equation," *Amer. Math. Soc. Trans.*, Vol. 35 (1933), pp. 785–791.

ON THE JOINT DISTRIBUTION OF THE MEDIANS IN SAMPLES FROM A MULTIVARIATE POPULATION

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It is well known [1] that in the case of a population having a single variate distributed according to a density function satisfying certain general conditions, the median of a sample is asymptotically normally distributed about the population median as a mean. It is the purpose of this paper to extend this result to populations involving more than one variate. Besides the theoretical interest of such a result, there may be some practical value in it when one is dealing with samples from a population for which the median is a more efficient statistic than the mean, as, for example, when the population variance is not finite.

The complexity of the exact distribution of the sample median increases rapidly with the number of variates which describe the population; it is almost impossible to write out completely the distribution for the general case of k variates. For this reason the author has chosen to give first a detailed presentation for the case of two variates, then use a condensed notation to establish the general result. This is a circuitous route, but it seems to be the only feasible one. A condensed notation is necessary for the general case, but presented alone it would be well-nigh incomprehensible.

1. Distribution of the median in two dimensions. An extension of A. T. Craig's [2] geometrical argument will be used to obtain the exact distribution of the sample median. Let us consider two variates x_1 and x_2 with density function $f(x_1, x_2)$ which shall satisfy the following conditions:

1. $f(x_1, x_2) \geq 0$
2. $$\int_{-\infty}^{\infty} f\left(x_1, \frac{1}{N}\right) dx_1 = \int_{-\infty}^{\infty} f(x_1, 0)\, dx_1 + O\left(\frac{1}{N}\right)$$
   $$\int_{-\infty}^{\infty} f\left(\frac{1}{N}, x_2\right) dx_2 = \int_{-\infty}^{\infty} f(0, x_2)\; dx_2 + O\left(\frac{1}{N}\right)$$
3. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 = 1$$
4. Each of the equations
   $$\int_{-\infty}^{\xi}\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 dx_1 = \tfrac{1}{2}$$
   $$\int_{-\infty}^{\xi}\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 dx_2 = \tfrac{1}{2}$$
   has a unique real root.

If ξ_1 and ξ_2 are the respective roots of the two equations of this last condition then the point (ξ_1, ξ_2) is defined to be the population median. It will be assumed in what follows that the coordinate system has been so chosen that $\xi_1 = 0 = \xi_2$.

Let a sample of $2n + 1$ elements $(x_{1\alpha}, x_{2\alpha})(\alpha = 1, 2, \cdots, 2n + 1)$ be drawn from this population. The sample median $(\tilde{x}_1, \tilde{x}_2)$ will be defined as an element (not necessarily in the sample) whose x_1 coordinate is the middle, with respect to magnitude, number of the set of numbers $x_{1\alpha}$, and whose x_2 coordinate is the middle number of the set of numbers $x_{2\alpha}$. Now let us compute the probability that the sample median will lie in the rectangle

$$\tilde{x}_i - \tfrac{1}{2}\, d\tilde{x}_i < x_i < \tilde{x}_i + \tfrac{1}{2}\, d\tilde{x}_i \qquad i = 1, 2.$$

This rectangle will be denoted by R''. The remainder of the plane will be divided into eight other regions $R_1, \cdots, R_4'$ as indicated by the dotted lines in Figure 1. The probability that an element will fall in the region $R_i^{(j)}$ will be denoted by

$$p_i^{(j)} = \int\int_{R_i^{(j)}} f(x_1, x_2)\, dx_1 dx_2 .$$

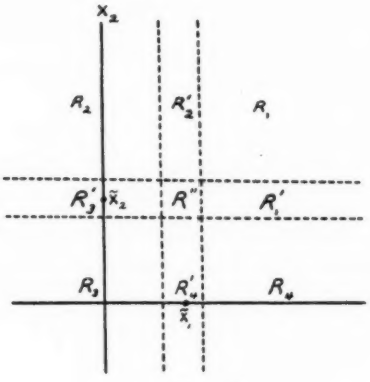

FIG. 1

Neglecting terms involving differentials of higher order we have

$$(1) \qquad \begin{aligned} p_1 &= \int_{\tilde{x}_1}^{\infty} \int_{\tilde{x}_2}^{\infty} f(x_1, x_2)\, dx_2 dx_1 \\ p_2 &= \int_{-\infty}^{\tilde{x}_1} \int_{\tilde{x}_2}^{\infty} f(x_1, x_2)\, dx_2\, dx_1 \\ &\vdots \\ p' &= \int_{\tilde{x}_1}^{\infty} f(x_1, \tilde{x}_2)\, dx_1\, d\tilde{x}_2 \\ &\vdots \\ p'' &= f(\tilde{x}_1, \tilde{x}_2)\, d\tilde{x}_1 d\tilde{x}_2 . \end{aligned}$$

We shall consider now that the sample is drawn from a multinomial population with probabilities $p_1, \cdots, p''$ and pick out those terms which give rise to a sample median in R''. If the median is an element of the sample, then that element must fall in R'' and the other elements must fall in the regions R_1, R_2, R_3, and R_4 in such a manner that

$$n_1 + n_2 = n_3 + n_4 = n$$

$$n_1 + n_4 = n_2 + n_3 = n$$

or so that

$$n_1 = n_3 \text{ and } n_2 = n_4 \tag{2}$$

where n_i is the number of elements in R_i. The probability that this occurs is

$$\sum_{n_1+n_2=n} \frac{(2n+1)!}{n_1!^2\, n_2!^2}\, p'' p_1^{n_1} p_2^{n_2} p_3^{n_1} p_4^{n_2} \tag{3}$$

Now suppose the median is determined by two different elements of the sample, for example one in R_1' and one in R_2', then there must be n_1 elements in R_1, $n_1 + 1$ elements in R_3, and n_2 elements in each of R_2 and R_4 with

$$n_1 + n_2 = n - 1. \tag{4}$$

The probability in this case is

$$p_1' p_2' \sum_{n_1+n_2=n-1} \frac{(2n+1)!}{n_1!\,(n_1+1)!\,n_2!^2}\, p_1^{n_1} p_2^{n_2} p_3^{n_1+1} p_4^{n_2}. \tag{5}$$

Continuing in this manner we obtain the distribution of the median, and letting $D(\tilde{x}_1, \tilde{x}_2)$ represent the density function giving this distribution we have

$$\begin{aligned} D(\tilde{x}_1, \tilde{x}_2)\, d\tilde{x}_1\, d\tilde{x}_2 = {} & p'' \Sigma \frac{(2n+1)!}{n_1!^2\, n_2!^2} (p_1 p_3)^{n_1} (p_2 p_4)^{n_2} \\ & + (p_3 p_1' p_2' + p_1 p_3' p_4') \Sigma \frac{(2n+1)!}{n_1!\,(n_1+1)!\,n_2!^2} (p_1 p_3)^{n_1} (p_2 p_4)^{n_2} \\ & + (p_2 p_1' p_4' + p_4 p_2' p_3') \Sigma \frac{(2n+1)!}{n_1!^2\, n_2!\,(n_2+1)!} (p_1 p_3)^{n_1} (p_2 p_4)^{n_2}. \end{aligned} \tag{6}$$

2. Asymptotic distribution of the median in two dimensions. As a simple notation

$$A = B(1 + O(1/\sqrt{n}))$$

will be abbreviated to read

$$A =\cdot B, \tag{7}$$

the dot after the equality sign indicating the omission of the factor $1 + O(1/\sqrt{n})$.

As is customary, the second term of this factor represents any function such that

$$\lim_{N\to\infty} NO(1/N) = L < \infty.$$

In order to get an approximation to (6) for large n we shall use the normal approximation for the multinomial distribution and compute the sums (these cannot be put in finite form) by integration. We use then the well-known result

$$\frac{m!}{\prod_1^r m_i!} \prod_1^r p_i^{m_i} = \cdot\, [A/(2\pi)^{r-1}]^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\sum_1^{r-1} A_{ij} z_i z_j\right) \prod_1^{r-1} dz_i, \tag{8}$$

where

$$z_i = (m_i - mp_i)/\sqrt{m}, \qquad i = 1, 2, \cdots, r-1, \tag{9}$$

$$A_{ii} = \frac{1}{p_i} + \frac{1}{p_r}, \qquad A_{ij} = \frac{1}{p_r}. \tag{10}$$

Returning to (6) it is to be noted that the fraction immediately following Σ in the first sum has one more factor in the denominator than the corresponding fractions in the other sums. This first sum may therefore be neglected in the asymptotic form as it is of order $1/n$ in comparison with the others. We consider now the second sum in (6) and let it be represented by the letter S

$$S = 2n(2n+1)p_1' p_2' \sum_{n_1+n_2=n-1} \frac{(2n-1)}{n_1!\,(n_1+1)!\,n_2!^2} p_1^{n_1} p_2^{n_2} p_3^{n_1+1} p_4^{n_2}. \tag{11}$$

Employing (8) and omitting certain terms of order $1/n$ we have

$$S = \cdot\, 4n^2 p_1' p_2' \Sigma\, [A/(2\pi)^3]^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\sum_1^3 A_{ij} z_i z_j\right) dz_1\, dz_2\, dz_3, \tag{12}$$

in which the A_{ij} are defined by (10) with $r = 4$, and

$$z_i = (n_i - 2np_i)/\sqrt{2n}, \qquad i = 1, 2, 3. \tag{13}$$

In view of the relations (2) between the n_i we have

$$\begin{aligned} z_2 &= \sqrt{2n}\,(\tfrac{1}{2} - p_1 - p_2) - z_1 = u_1 - z_1 \\ z_3 &= \sqrt{2n}\,(p_1 - p_3) - z_1 = u_2 - z_1, \end{aligned} \tag{14}$$

in which relations we have defined the new symbols u_1 and u_2. It will be recalled that in (8) the factors dz_i correspond to factors $1/\sqrt{m}$, we therefore let dz_2 and dz_3 in (12) cancel a factor $2n$ from the coefficient of the exponential, and after substituting (14) in (12) find that

$$\begin{aligned} S = \cdot\, 2np_1' p_2' \Sigma\, [A/(2\pi)^3]^{\frac{1}{2}} \exp\Big\{-\tfrac{1}{2}\Big[& z_1^2\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right) \\ & + 2z_1\left(\frac{u_1+u_2}{p_4} - \frac{u_1}{p_2} + \frac{u_2}{p_3}\right) + \frac{(u_1+u_2)^2}{p_4} + \frac{u_1^2}{p_2} + \frac{u_2^2}{p_3}\Big]\Big\}\, dz_1. \end{aligned} \tag{15}$$

The summation can now be performed to within terms of order $1/\sqrt{n}$ by integration with respect to z_1 between the limits $-\infty$ and $+\infty$; this gives us

$$S = \cdot \frac{2np_1'p_2'}{2\pi} A^{\frac{1}{2}} \Big/ \left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}\right)^{\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\left[\frac{(u_1+u_2)^2}{p_4}+\frac{u_1^2}{p_2}\right.\right.$$
$$\left.\left.+\frac{u_2^2}{p_3}-\left(\frac{u_1+u_2}{p_4}-\frac{u_1}{p_2}+\frac{u_2}{p_3}\right)^2\Big/\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}\right)\right]\right\}. \tag{16}$$

At this point some new symbols are required. We let q_i and q_i' represent the results of replacing $\tilde{x}_1$ and $\tilde{x}_2$ by zero in the integrals of the relations (1)

$$\begin{aligned}
q_1 &= \int_0^\infty \int_0^\infty f(x_1, x_2)\, dx_1\, dx_2 & q_1' &= \int_0^\infty f(x_1, 0)\, dx_1 \\
q_2 &= \int_{-\infty}^0 \int_0^\infty f(x_1, x_2)\, dx_1\, dx_2 & q_2' &= \int_0^\infty f(0, x_2)\, dx_2 \\
q_3 &= \int_{-\infty}^0 \int_{-\infty}^0 f(x_1, x_2)\, dx_1\, dx_2 & q_3' &= \int_{-\infty}^0 f(x_1, 0)\, dx_1 \\
q_4 &= \int_0^\infty \int_{-\infty}^0 f(x_1, x_2)\, dx_1\, dx_2 & q_4' &= \int_{-\infty}^0 f(0, x_2)\, dx_2
\end{aligned} \tag{17}$$

then

$$q_1 + q_2 = q_3 + q_4 = q_1 + q_4 = q_2 + q_3 = \tfrac{1}{2} \tag{18}$$

and

$$q_1 = q_3\,, \qquad q_2 = q_4\,. \tag{19}$$

Also we let

$$a_1 = q_2' + q_3'\,, \qquad a_2 = q_1' + q_4'\,, \tag{20}$$

$$y_1 = \sqrt{2n}\, a_1\tilde{x}_1\,, \qquad y_2 = \sqrt{2n}\, a_2\tilde{x}_2\,. \tag{21}$$

We have now

$$\begin{aligned}
p_i &= \cdot\, q_i\,, & i &= 1, 2, 3, 4, \\
p_i' &= \cdot\, q_i' d\tilde{x}_2\,, & i &= 1, 3, \\
p_i' &= \cdot\, q_i' d\tilde{x}_1\,, & i &= 2, 4.
\end{aligned} \tag{22}$$

Also

$$\begin{aligned}
u_1 &= \sqrt{2n}\,(\tfrac{1}{2} - p_1 - p_2) \\
&= \sqrt{2n} \int_{-\infty}^{\infty} \int_0^{\tilde{x}_2} f(x_1, x_2)\, dx_1 dx_2
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2n}\,\tilde{x}_2 \int_{-\infty}^{\infty} f(x_1, \theta\tilde{x}_2)\,dx_1, && 0 \le \theta \le 1,\\
&=\cdot \sqrt{2n}\,\tilde{x}_2 \int_{-\infty}^{\infty} f(x_1, 0)\,dx_1\\
&=\cdot \sqrt{2n}\,a_2\tilde{x}_2\\
&=\cdot y_2.
\end{aligned} \tag{23}
$$

Similarly

$$
\begin{aligned}
u_2 &= \sqrt{2n}\,(p_1 - p_2)\\
&=\cdot -(y_1 + y_2).
\end{aligned} \tag{24}
$$

The result of substituting (22), (23) and (24) in (16) with some further simplification using (18) and (19) is

$$
S =\cdot \frac{2nq_1'q_2'}{2\pi\sqrt{q_1q_2}} \exp\left(-\frac{1}{2}\frac{y_1^2 - 4(q_1 - q_2)y_1y_2 + y_2^2}{4q_1q_2}\right) d\tilde{x}_1\,d\tilde{x}_2. \tag{25}
$$

The other three sums of (6) will give rise to the same expression except that the factors $q_1'q_2'$ will be different; it is clear then that

$$
\begin{aligned}
D(\tilde{x}_1, \tilde{x}_2)\,d\tilde{x}_1\,d\tilde{x}_2 &=\cdot \frac{2n(q_1'q_2' + q_1'q_2' + q_2'q_3' + q_3'q_4')}{2\pi\sqrt{q_1q_2}}\\
&\qquad \times \exp\left(-\frac{1}{2}\frac{y_1^2 - 4(q_1 - q_2)y_1y_2 + y_2^2}{4q_1q_2}\right) d\tilde{x}_1\,d\tilde{x}_2
\end{aligned}
$$

$$
=\cdot \frac{2na_1a_2}{2\pi\sqrt{q_1q_2}} \exp\left(-n\frac{a_1^2\tilde{x}_1^2 - 4(q_1 - q_2)a_1a_2\tilde{x}_1\tilde{x}_2 + a_2^2\tilde{x}_2^2}{4q_1q_2}\right) d\tilde{x}_1\,d\tilde{x}_2, \tag{26}
$$

$$
=\cdot \frac{1}{2\pi\sqrt{q_1q_2}} \exp\left(-\frac{1}{2}\frac{y_1^2 - 4(q_1 - q_2)y_1y_2 + y_2^2}{4q_1q_2}\right) dy_1\,dy_2. \tag{27}
$$

This is the asymptotic form for the distribution of the median in two dimensions.

3. Distribution of the median in k dimensions. We consider now a population characterized by a density function $f(x_1, \cdots, x_k)$ defined over a euclidean space of k dimensions satisfying conditions like those required of $f(x_1, x_2)$ in section 1, and we assume that the population median is at the origin so that the integral of the density function over any half-space determined by a coordinate hyperplane is $\frac{1}{2}$.

A sample of $2n + 1$ elements will have a median $(\tilde{x}_1, \cdots, \tilde{x}_k)$ each coordinate of which is the middle number of the set of numbers giving the corresponding coordinate of the elements of the sample. To obtain the probability that the sample median lies in the hyperparallopiped $\tilde{x}_\alpha - \frac{1}{2}d\tilde{x}_\alpha < x_\alpha < \tilde{x}_\alpha + \frac{1}{2}d\tilde{x}_\alpha$ $(\alpha = 1, 2, \cdots, k)$, we divide the space into 3^k regions by means of hyperplanes

perpendicular to the coordinate axes through the points $\tilde{x}_\alpha \pm \frac{1}{2} d\tilde{x}_\alpha$ on the coordinate axes. These regions are illustrated in Figure 2 for the case of three dimensions. The coordinate axes have been omitted in this figure. There will be 2^k primary regions denoted by $R_1, R_2, \cdots, R_{2^k}$ corresponding to the octants of the figure; $k2^{k-1}$ regions with one differential dimension denoted by $R_1', R_2', \cdots, R_{k2^{k-1}}'$ corresponding to the quarter slabs of the figure; $\binom{k}{2}2^{k-1}$ regions with two differential dimensions corresponding to the half strips of the figure, and so forth. Probabilities associated with these regions are defined by

$$p_i^{(j)} = \int_{R_i^{(j)}} f(x_1, \cdots, x_k)\, dx_1 \cdots dx_k .$$

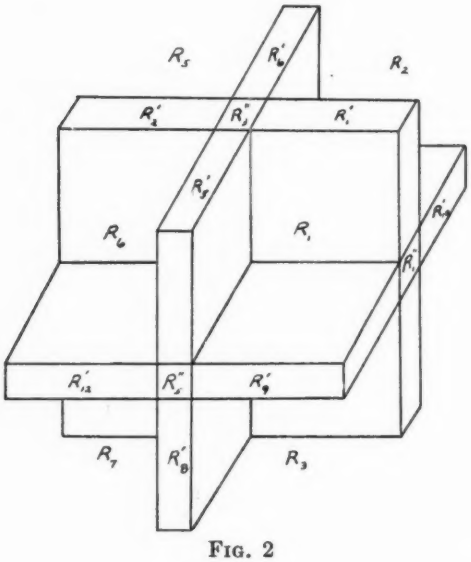

FIG. 2

If the sample median is determined by k different elements of the sample there will be one of these k elements in each of k regions R_i' whose differential dimensions are mutually orthogonal and the other elements of the sample will fall in the regions R_i in such a way that n elements of the sample will lie on either side of any of the k hyperplanes $x_\alpha = \tilde{x}_\alpha$. The probability of this occurrence for a particular choice of k of the regions R_i' is

$$S = \prod_{\alpha=1}^{k} p_{i_\alpha}' \sum \frac{(2n+1)!}{\prod n_i!} \prod_{i=1}^{2^k} p_i^{n_i} \tag{28}$$

in which the 2^k indices n_i are subject to k independent restrictions of the type

$$\Sigma' n_i = n - c_\alpha , \tag{29}$$

where c_α is an integer such that $0 \leq c_\alpha < k$, and the prime on Σ indicates that the sum is to be taken over all n_i on one side of a hyperplane $x_\alpha = \bar{x}_\alpha$. n_i is the number of elements in R_i and besides the k conditions (29) we have also

$$\sum_1^{2^k} n_i = 2n - k + 1. \tag{30}$$

In order to include all ways in which the median is determined by k different elements of the sample we must add together $2^{k(k-1)}$ sums of the type (28). If the median is determined by less than k elements, say $k - h$ elements, then the fraction $(2n + 1)!/\Pi n_i!$ will have h extra factors in the denominator and hence the sum will be of order $1/n^h$ as compared with that of (28) and may be neglected in obtaining an asymptotic expression.

Thus we need only find the limiting form of (28)

$$S = (2n + 1)(2n) \cdots (2n - k + 2) \prod_1^k p'^{}_{i_\alpha} \sum \frac{(2n - k + 1)!}{\prod n_i!} \prod_1^{2^k} p_i^{n_i},$$

which after substituting (8) and neglecting terms of lower order becomes

$$S = \cdot (2n)^k \prod p'^{}_{i_\alpha} \sum (A/(2\pi)^{2^k-1})^{\frac{1}{2}} \exp\left(-\tfrac{1}{2} \sum A_{ij} z_i z_j\right) \prod_1^{2^k-1} dz_i, \tag{31}$$

in which the A_{ij} are defined by (10) with $r = 2^k$ and

$$z_i = (n_i - 2np_i)/\sqrt{2n}, \qquad i = 1, 2, \cdots, 2^k - 1. \tag{32}$$

Now we define

$$u_\alpha = \sqrt{2n}(\tfrac{1}{2} - \Sigma' p_i), \qquad \alpha = 1, 2, \cdots, k, \tag{33}$$

the Σ' having the same significance as in (29). These conditions (29) may now be put in the form

$$z_\alpha = \cdot u_\alpha - L_\alpha(z),$$

in which $L_\alpha(z)$ is a sum of a certain subset of the variables $z_{k+1}, \cdots, z_{2^k-1}$. Care must be taken in labeling the regions R_i in order to be able to solve for $z_1, \cdots, z_k$ in this form. After substituting these relations in (31) we replace $\prod_1^k dz_\alpha$ by $(1/2n)^{k/2}$ and perform the summation to within terms of order $1/\sqrt{n}$ by integrating the remaining z_i from $-\infty$ to $+\infty$; the result is

$$S = \cdot (2n/2\pi)^{k/2} \prod_{\alpha=1}^k p'^{}_{i_\alpha} \sqrt{B} \exp\left(-\tfrac{1}{2} \sum_1^k B_{\alpha\beta} u_\alpha u_\beta\right), \tag{34}$$

in which the $B_{\alpha\beta}$ are functions of the p_i, and $B = |B_{\alpha\beta}|$. As in (17) and (20)

we define

$$q_i = \int_{\bar{R}_i} f(x_1, \cdots, x_k)\Pi\, dx_\alpha$$

$$q_i' = \int_{\bar{R}_i'} f(x_1, \cdots, x_k)\Pi'\, dx_\alpha \tag{35}$$

$$a_\alpha = \int_{x_\alpha=0} f(x_1, \cdots, x_k)\Pi'\, dx_\alpha = \Sigma' q_i',$$

in which $\bar{R}_i$ is the set of regions bounded by the coordinate hyperplanes R_i' are regions into which the coordinate hyperplanes are divided by the remaining coordinate hyperplanes. Π' indicates that one of the differentials is omitted and the variate corresponding to that differential is put equal to zero in $f(x_1, \cdots, x_k)$; Σ' indicates the sum over all q' determined by regions lying in the hyperplane $x_\alpha = 0$. It is clear that

$$p_i = \cdot q_i$$

$$\prod_\alpha p_{i_\alpha}' = \cdot \prod_\alpha q_{i_\alpha}' d\tilde{x}_\alpha \tag{36}$$

$$u_\alpha = \cdot \sqrt{2n} \sum_{\beta=1}^{k} \delta_{\alpha\beta} a_\beta \tilde{x}_\beta = \sum \delta_{\alpha\beta} y_\beta,$$

where

$$\delta_{\alpha\beta} = \pm 1 \text{ or } 0, \qquad \text{and} \qquad \gamma_\beta = \sqrt{2n} a_\beta \tilde{x}_\beta .$$

Making these substitutions in (34) we have

$$S = \cdot (2n/2\pi)^{k/2} \prod_1^k q_{i_\alpha}' \sqrt{C} \exp\left(-n \sum_1^k C_{\alpha\beta} a_\alpha a_\beta \tilde{x}_\alpha \tilde{x}_\beta\right) \prod d\tilde{x}_\alpha, \tag{37}$$

and adding together all possible sums of the type (28) we have the asymptotic form of the distribution of the sample median

$$D(\tilde{x}_1, \cdots, \tilde{x}_k) \prod d\tilde{x}_\alpha \tag{38}$$

$$= \cdot (2n/2\pi)^{k/2} \prod_1^k a_\alpha \sqrt{C} \exp\left(-n \sum_1^k C_{\alpha\beta} a_\alpha a_\beta \tilde{x}_\alpha \tilde{x}_\beta\right) \prod d\tilde{x}_\alpha$$

$$= \cdot (1/2\pi)^{k/2} \sqrt{C} \exp\left(-\tfrac{1}{2} \sum C_{\alpha\beta} y_\alpha y_\beta\right) \prod dy_\alpha, \tag{39}$$

in which the $C_{\alpha\beta}$ are functions of the q_i.

4. The case of three dimensions. The computation of the coefficients $C_{\alpha\beta}$ of (39) requires the evaluation of a determinant of order $2^k - k$ for each one of them. This work was quite laborious even for $k = 3$ and the author made no attempt to find their explicit expression for larger values of k.

If we let a subscript $+$ indicate integration of the density function $f(x_1, x_2, x_3)$ from 0 to ∞, and a subscript—indicate integration from $-\infty$ to 0,

as for example,

$$f_{++-} = \int_0^\infty \int_0^\infty \int_{-\infty}^0 f(x_1, x_2, x_3)\, dx_3\, dx_2\, dx_1,$$

then the q_i of (35) will be defined as follows

$$\begin{aligned} q_1 &= f_{+++} & q_5 &= f_{-++} \\ q_2 &= f_{++-} & q_6 &= f_{-+-} \\ q_3 &= f_{+-+} & q_7 &= f_{--+} \\ q_4 &= f_{+--} & q_8 &= f_{---} \end{aligned} \tag{40}$$

The coefficients $C_{\alpha\beta}$ may be written

$$\begin{aligned} DC_{11} &= 2(q_1 + q_5)(q_2 + q_6) \\ DC_{22} &= 2(q_1 + q_3)(q_2 + q_4) \\ DC_{33} &= 2(q_1 + q_2)(q_3 + q_4) \\ DC_{12} &= q_3q_5 + q_4q_6 - q_1q_7 - q_2q_8 \\ DC_{13} &= q_2q_5 + q_4q_7 - q_1q_6 - q_3q_8 \\ DC_{23} &= q_2q_3 + q_4q_7 - q_1q_4 - q_5q_8, \end{aligned} \tag{41}$$

where

$$\begin{aligned} D = q_1q_2q_3q_4\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4}\right) + q_5q_6q_7q_8\left(\frac{1}{q_5} + \frac{1}{q_6} + \frac{1}{q_7} + \frac{1}{q_8}\right) \\ + 2(q_5 + q_6)(q_7 + q_8)(q_1q_2 + q_3q_4) \\ + 2(q_5 + q_7)(q_6 + q_8)(q_1q_3 + q_2q_4) \\ + 2(q_5 + q_8)(q_6 + q_7)(q_1q_4 + q_2q_3) \\ + 8(q_1q_4q_6q_7 + q_2q_3q_5q_8) \end{aligned} \tag{42}$$

(41) and (42) can of course be put in different forms by using the four relations between the q_i. The a_α of (38) are defined in (35); for $k = 3$ they are

$$\begin{aligned} a_1 &= \int_{-\infty}^\infty \int_{-\infty}^\infty f(0, x_2, x_3)\, dx_2\, dx_3 \\ a_2 &= \int_{-\infty}^\infty \int_{-\infty}^\infty f(x_1, 0, x_3)\, dx_1\, dx_3 \\ a_3 &= \int_{-\infty}^\infty \int_{-\infty}^\infty f(x_1, x_2, 0)\, dx_1\, dx_2. \end{aligned} \tag{43}$$

5. The normal distribution in two dimensions. If the density function of the second section of the paper is normal

$$(44)\qquad f(x_1, x_2) = 1/(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}) \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x_1^2}{\sigma_1^2} - 2\rho\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)\right],$$

we find that the parameters of (26) are

$$(45)\qquad \begin{aligned} q_1 &= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho, & q_2 &= \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho, \\ a_1 &= \frac{1}{\sqrt{2\pi}\,\sigma_1}, & a_2 &= \frac{1}{\sqrt{2\pi}\,\sigma_2}. \end{aligned}$$

These give an interesting result—the correlation coefficient of the asymptotic distribution of the sample medians is

$$(46)\qquad \rho_m = \frac{2}{\pi}\sin^{-1}\rho$$

hence

$$(47)\qquad |\rho_m| \leq |\rho|$$

the equality sign holding only when $\rho = 0$ or ± 1.

REFERENCES

[1] S. S. Wilks, *Statistical Inference*, Ann Arbor, Edwards Bros., 1937.

[2] A. T. Craig, "On the distributions of certain statistics," *Amer. Jour. Math.*, Vol. 54 (1932), pp. 353–366.

SAMPLES FROM TWO BIVARIATE NORMAL POPULATIONS[1]

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1. **Introduction.** In multivariate analysis involving p variates, or in analysis of variance of m samples from univariate populations, we are often interested in the hypothesis of the equality of variances; viz., that

$$\sigma_1 = \sigma_2 = \cdots = \sigma_p , \qquad \text{in the case of } p \text{ variates;}$$

or

$$\sigma_1 = \sigma_2 = \cdots = \sigma_m , \qquad \text{in the case of } m \text{ samples.}$$

As a matter of fact, it seldom occurs that these hypotheses are true, but the ratio between the variances might be known.

Hotelling [5] has suggested that if

$$\sigma_1^2/k_1 = \sigma_2^2/k_2 = \cdots = \sigma_m^2/k_m = \sigma^2,$$

where the k's are known constants, we can apply the transformation

$$\begin{aligned} x_1' &= w_1 x_1 , \\ x_2' &= w_2 x_2 , \\ &\cdots\cdots\cdots \\ x_m' &= w_m x_m , \end{aligned}$$

where

$$w\sqrt{k_1} = w_2\sqrt{k_2} = \cdots = w_m\sqrt{k_m} = 1,$$

so that after transformation the variances become equal, i.e.,

$$\sigma_1' = \sigma_2' = \cdots = \sigma_m' ,$$

and the required analysis can be carried out. This method is similarly applicable in the multivariate case.

In a previous paper [7], I developed a series of hypotheses concerning samples from a bivariate normal population under the assumption that

$$\sigma_1 = \sigma_2 .$$

In case $\sigma_1^2/k_1 = \sigma_2^2/k_2$, where k_1 and k_2 are two distinct known constants, similar results may be obtained by the use of the transformation $x_1' = w_1 x_1$; $x_2' = w_2 x_2$; where $w_1\sqrt{k_1} = w_2\sqrt{k_2} = 1$.

[1] Presented to the American Mathematical Society at Washington, D. C., May 3, 1941.

In multivariate analysis, the hypotheses usually of interest concerning correlation coefficients may be classified in two categories, viz.,

(i) that the correlation coefficient is equal to a specified value, e.g., in simple correlation $\rho_{12} = \rho_0$, in partial correlation, $\rho_{12 \cdot 3} = \rho_0$, in multiple correlation, $\rho_{1 \cdot 23} = \rho_0$, or in correlation between two sets of variates [4][2], $Q = Q_0$; of special interest is the hypothesis of the vanishing of such correlation coefficients.

(ii) that two given correlation coefficients are equal, e.g., (1) correlation coefficients ρ_1 and ρ_2 in the correlation matrix of a multivariate distribution are equal (Hotelling [6]), or (2) the correlation coefficients ρ_{12} and ρ'_{12} in two bivariate populations are equal.

R. A. Fisher in his earlier paper [3] introduced the transformation $z = \frac{1}{2} \log \frac{1+r}{1-r}$ which provides a very satisfactory, though approximate, method for the comparison of two correlation coefficients. Brander [1] treated the same problem by the method of the likelihood ratio criterion.

The present paper is an attempt to obtain different criteria by the likelihood ratio method (Neyman and Pearson [9], [10], [11]) for testing, by means of samples, the equality of correlation coefficients in two bivariate normal populations under the following sets of conditions: (1) $\sigma_1 = \sigma_2$ and $\sigma'_1 = \sigma'_2$; (2) $\sigma_1 = \sigma_2$, $\xi_1 = \xi_2$ and $\sigma'_1 = \sigma'_2$, $\xi'_1 = \xi'_2$. The results may be extended to the cases (3) $\sigma_1^2/k_1 = \sigma_2^2/k_2$ and $\sigma_1'^2/k'_1 = \sigma_2'^2/k'_2$; (4) $\sigma_1^2/k_1 = \sigma_2^2/k_2$, $\xi_1^2/k_1 = \xi_2^2/k_2$ and $\sigma_1'^2/k'_1 = \sigma_2'^2/k'_2$, $\xi_1'^2/k'_1 = \xi_2'^2/k'_2$, where the k's are known constants.

2. The hypotheses. Two samples, each being of two variates (x_1, x_2) and (x'_1, x'_2), of size N and N', are supposed to be drawn at random, respectively, from two independent normal bivariate populations, with the following distributions:

$$(1) \quad \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\xi_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\xi_1}{\sigma_1}\right)\left(\frac{x_2-\xi_2}{\sigma_2}\right) + \left(\frac{x_2-\xi_2}{\sigma_2}\right)^2\right]\right\},$$

$$(2) \quad \frac{1}{2\pi\sigma'_1\sigma'_2\sqrt{1-\rho'^2}} \exp\left\{-\frac{1}{2(1-\rho'^2)}\left[\left(\frac{x'_1-\xi'_1}{\sigma'_1}\right)^2 - 2\rho'\left(\frac{x'_1-\xi'_1}{\sigma'_1}\right)\left(\frac{x'_2-\xi'_2}{\sigma'_2}\right) + \left(\frac{x'_2-\xi'_2}{\sigma'_2}\right)^2\right]\right\},$$

where $\xi_1, \xi_2, \sigma_1, \sigma_2, \rho; \xi'_1, \xi'_2, \sigma'_1, \sigma'_2, \rho'$ are the unknown parameters of the populations.

The hypotheses to be considered in the present paper are:

H_1: Assuming $\sigma_1 = \sigma_2$ and $\sigma'_1 = \sigma'_2$, to test $\rho = \rho'$.

H_2: Assuming $\sigma_1 = \sigma_2$, $\xi_1 = \xi_2$, and $\sigma'_1 = \sigma'_2$, $\xi'_1 = \xi'_2$, to test $\rho = \rho'$.

[2] See bibliography at the end of the paper.

The derivation and the distribution of the criteria for testing these hypotheses may be simplified by the following simultaneous transformations:

$$X = \frac{1}{\sqrt{2}}(x_1 - x_2) \qquad Y = \frac{1}{\sqrt{2}}(x_1 + x_2) \tag{3}$$

$$X' = \frac{1}{\sqrt{2}}(x_1' - x_2') \qquad Y' = \frac{1}{\sqrt{2}}(x_1' + x_2') \tag{4}$$

The corresponding normal bivariate distributions in the transformed variables (X, Y) and (X', Y') are obtained, viz.

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\left(\frac{X-\xi}{\sigma_X}\right)^2 - 2\rho_{XY}\left(\frac{X-\xi}{\sigma_X}\right)\left(\frac{Y-\eta}{\sigma_Y}\right) + \left(\frac{Y-\eta}{\sigma_Y}\right)^2\right]\right\} dX\,dY, \tag{5}$$

$$\frac{1}{2\pi\sigma_X'\sigma_Y'\sqrt{1-\rho_{XY}'^2}} \exp\left\{-\frac{1}{2(1-\rho_{XY}'^2)}\left[\left(\frac{X'-\xi'}{\sigma_X'}\right)^2 - 2\rho_{XY}'\left(\frac{X'-\xi'}{\sigma_X'}\right)\left(\frac{Y'-\eta'}{\sigma_Y'}\right) + \left(\frac{Y'-\eta'}{\sigma_Y''}\right)^2\right]\right\} dX'\,dY'. \tag{6}$$

The conditions corresponding to

$$\sigma_1 = \sigma_2 \quad \text{and} \quad \sigma_1' = \sigma_2', \tag{7}$$

are that

$$\rho_{XY} = 0 \quad \text{and} \quad \rho_{XY}' = 0. \tag{8}$$

Also, for a given ρ and ρ', we have from (7)

$$\sigma_Y^2 = \gamma\sigma_X^2 \quad \text{and} \quad \sigma_Y'^2 = \gamma'\sigma_X'^2, \tag{9}$$

where

$$\gamma = \frac{1+\rho}{1-\rho} \qquad \text{and} \qquad \gamma' = \frac{1+\rho'}{1-\rho'}. \tag{10}$$

Following the notation of (9) and (10), the hypotheses H_1' and H_2' corresponding to H_1 and H_2 are:

H_1': Assuming $\rho_{XY} = 0$, and $\rho_{XY}' = 0$, to test $\gamma = \gamma'$.

H_2': Assuming $\rho_{XY} = 0$, $\xi = 0$, and $\rho_{XY}' = 0, \xi' = 0$, to test $\gamma = \gamma'$.

3. **The derivation of the criteria.** Let $(x_{1i}, x_{2i})(x_{1j}', x_{2j}')$ be the measurements of the characters on the ith and jth individuals in the two samples from their respective populations. After transformation, the corresponding measurements become (X_i, Y_i) and (X_j', Y_j'). Let $p(E)$ denote the joint elementary proba-

bility law of the N and N' observations, $E = (X_1, \cdots, X_N, Y_1, \cdots, Y_N; X'_1, \cdots, X'_{N'}, Y', \cdots, Y'_{N'})$.

Following Neyman and Pearson, we shall use Ω to designate the class of admissible populations under conditions which can be assumed to be satisfied in any case; and ω to designate a subclass of Ω under conditions which are satisfied only if the hypothesis to be tested is true.

Thus for H', Ω specifies for $\rho_{XY} = \rho'_{XY} = 0$, any real values of ξ, η, ξ', η' and any positive values of σ_X, σ_Y, σ'_X, σ'_Y; ω specifies $\rho_{XY} = \rho'_{XY} = 0$, any real values of ξ, η, ξ', η' and any positive values of σ_Y and γ which are defined by (9). While for H', Ω specifies $\rho_{XY} = \rho'_{XY} = 0$, $\xi = \xi' = 0$, any real values of η and η' and any positive values of σ_X, σ_Y, σ'_X, σ'_Y; ω specifies $\rho_{XY} = \rho'_{XY} = 0$, $\xi = \xi' = 0$, any real values of η and η', and any positive values of σ_Y and γ which are defined by (9).

For our hypothesis H'_1, the values of the parameters required to make $p(\Omega)$ a maximum are:

$$\hat{\xi} = \bar{X}, \quad \hat{\eta} = \bar{Y}, \quad \hat{\sigma}_X = s_X, \quad \hat{\sigma}_Y = s_Y$$
$$\hat{\xi}' = \bar{X}', \quad \hat{\eta}' = \bar{Y}', \quad \hat{\sigma}'_X = s'_X, \quad \hat{\sigma}'_Y = s'_Y.$$

Thus $p(\Omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \frac{1}{s_X^N s_Y^N s_X'^{N'} s_Y'^{N'}} e^{-N-N'}$

To obtain $p(\omega \max)$, let us define, according to the notation in the writer's previous paper [7],

$$R_1 = \frac{2Y s_1 s_2}{s_1^2 + s_2^2} \quad \text{and} \quad R'_1 = \frac{2Y' s'_1 s'_2}{s_1'^2 + s_2'^2}$$

and

$$u = \frac{s_Y^2}{s_X^2} = \frac{1 + R_1}{1 - R_1} \qquad u' = \frac{s_Y'^2}{s_X'^2} = \frac{1 + R'_1}{1 - R'_1}.$$

Then the values making $p(\omega)$ a maximum are:

$$\hat{\xi} = \bar{X}, \quad \hat{\eta} = \bar{Y}, \quad \sigma_Y^2 = \tfrac{1}{2}s_X^2(\hat{\gamma} + u)$$
$$\hat{\xi}' = \bar{X}', \quad \hat{\eta}' = \bar{Y}', \quad \sigma_Y'^2 = \tfrac{1}{2}s_X'^2(\hat{\gamma} + u')$$

and $\hat{\gamma}$ is the positive root of the equation

$$(N + N')\gamma^2 - (N - N')(u - u')\gamma - (N + N')uu' = 0$$

or

$$\hat{\gamma} = \frac{(N - N')(u - u') + \sqrt{(N - N')^2(u - u')^2 + 4(N + N')uu'}}{2(N + N')} \tag{11}$$

$$= \gamma_1, \text{ say.}$$

Then

$$p(\omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \left[\frac{2\sqrt{\gamma_1}}{(\gamma_1+u)s_X^2}\right]^N \left[\frac{2\sqrt{\gamma_1}}{(\gamma_1+u')s_X'^2}\right]^{N'} e^{-N-N'},$$

and the likelihood ratio criterion for the hypothesis H_1' is

$$\lambda = \frac{p(\omega \max)}{p(\Omega \max)} = \left[\frac{2\sqrt{\gamma_1}\, s_Y}{(\gamma_1+u)s_X}\right]^N \left[\frac{2\sqrt{\gamma_1}\, s_Y'}{(\gamma_1+u')s_X}\right]^{N'} = \left[\frac{2\sqrt{\gamma_1 u}}{\gamma_1+u}\right]^N \left[\frac{2\sqrt{\gamma_1 u'}}{\gamma_1+u'}\right]^{N'}. \tag{12}$$

For H_2', the values the parameters to make $p(\omega)$ a maximum are:

$$\hat{\eta} = \bar{Y}, \qquad \hat{\sigma}_X^2 = \frac{1}{N}\Sigma X^2 \qquad \hat{\sigma}_Y = s_Y$$

$$\hat{\eta}' = \bar{Y}', \qquad \hat{\sigma}_X'^2 = \frac{1}{N'}\Sigma X'^2 \qquad \hat{\sigma}_Y' = s_Y'.$$

Thus

$$p(\Omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \frac{\sqrt{NN'}}{(\Sigma X^2)^{N/2}(\Sigma X'^2)^{N'/2} s_Y^N s_Y'^{N'}} e^{-N-N'}.$$

Similarly, if we write

$$R_2 = \frac{2\gamma s_1 s_2 - \frac{1}{2}(\bar{x}_1-\bar{x}_2)^2}{s_1^2+s_2^2+\frac{1}{2}(\bar{x}_1-\bar{x}_2)^2}, \qquad R_2' = \frac{2\gamma' s_1' s_2' - \frac{1}{2}(\bar{x}_1'-\bar{x}_2')^2}{s_1'^2+s_2'^2+\frac{1}{2}(\bar{x}_1'-\bar{x}_2')^2},$$

and

$$v = \frac{Ns_Y^2}{\Sigma X^2} = \frac{s_Y^2}{s_X^2+\bar{x}^2} = \frac{1+R_2}{1-R_2}, \qquad v' = \frac{Ns_Y'^2}{\Sigma X'^2} = \frac{1+R_2'}{1-R_2'},$$

the values to make $p(\omega)$ a maximum are:

$$\hat{\eta} = \bar{Y}, \qquad \sigma_Y^2 = \frac{1}{2N}\Sigma X^2(\hat{\gamma}+v)$$

$$\hat{\eta}' = \bar{Y}', \qquad \sigma_Y'^2 = \frac{1}{2N'}\Sigma X'^2(\hat{\gamma}+v)$$

$$\hat{\gamma} = \frac{(N-N')(v-v') + \sqrt{(N-N')^2(v-v')^2 + 4(N+N')^2 uu'}}{2(N+N')} \tag{13}$$

$$= \gamma_2, \text{ say.}$$

Then

$$p(\omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \left[\frac{2N\sqrt{\gamma_2}}{(\gamma_2+v)\Sigma X^2}\right]^N \left[\frac{2N'\sqrt{\gamma_2}}{(\gamma_2+v')\Sigma X'^2}\right]^{N'},$$

and the likelihood ratio criterion for the hypothesis H_2' is

$$\lambda_2 = \frac{p(\omega \max)}{p(\Omega \max)} = \left[\frac{2\sqrt{N\gamma_2}s_Y}{(\gamma_2+v)\sqrt{\Sigma X^2}}\right]^N\left[\frac{2\sqrt{N'\gamma_2}s_Y'}{(\gamma_2+v')\sqrt{\Sigma X'^2}}\right]^{N'} = \left[\frac{2\sqrt{\gamma_2 v}}{\gamma_2+v}\right]^N\left[\frac{2\sqrt{\gamma_2 v'}}{\gamma_2+v'}\right]^{N'}. \tag{14}$$

The case $N = N'$. The above criteria λ_1 and λ_2 cannot in general be expressed simply, but when $N = N'$, by (11) and (13)

$$\gamma_1 = \sqrt{uu'}, \qquad \gamma_2 = \sqrt{vv'},$$

and

$$\lambda_1 = \left[\frac{4\sqrt{uu'}}{(\sqrt{u}+\sqrt{u'})^2}\right]^N, \qquad \lambda_2 = \left[\frac{4\sqrt{vv'}}{(\sqrt{v}+\sqrt{v'})^2}\right]^N,$$

or we may express as monotonic functions of λ_1 and λ_2,

$$L_1 = \lambda_1^{2/(N+N')} = \lambda_1^{1/N} = \frac{4}{\left(\sqrt[4]{\frac{u}{u'}}+\sqrt[4]{\frac{u'}{u}}\right)^2}, \tag{15}$$

$$L_2 = \lambda_2^{1/N} = \frac{4}{\left(\sqrt[4]{\frac{v}{v'}}+\sqrt[4]{\frac{v'}{v}}\right)^2}. \tag{16}$$

Thus, λ's, L's, or their functions $\frac{u}{u'}$, $\frac{v}{v'}$, may be used as the criteria in the present case.

Furthermore, if we introduce,

$$z = \tfrac{1}{2}\log u, \quad \text{and} \quad z' = \tfrac{1}{2}\log u', \tag{17}$$

we have

$$\tfrac{1}{2}(z-z') = \tfrac{1}{4}\log\frac{u}{u'} \quad \text{or} \quad \sqrt[4]{\frac{u}{u'}} = e^{\frac{1}{2}(z-z')}.$$

Thus L_1 can be written in terms of z and z'

$$L_1 = 4/(e^{\frac{1}{2}(z-z')}+e^{-\frac{1}{2}(z-z')}) = 1/\cosh^2\tfrac{1}{2}(z-z') = \operatorname{sech}^2\tfrac{1}{2}(z-z'), \tag{18}$$

and $z - z' = w$, say, may be used also as a criterion for H_1.

We shall now proceed to obtain the distributions of some of these statistics.

4. **The distributions of u/u' and v/v'.** Since Ns_Y^2/σ_Y^2 and Ns_X^2/σ_X^2 have independently the χ^2 distribution with $N-1$ degrees of freedom,

$$u = \frac{s_Y^2}{s_X^2} = \frac{\sigma_Y^2\chi_2^2}{\sigma_X^2\chi_1^2} = \frac{\gamma\chi_2^2}{\chi_1^2}$$

and u/γ has the F distribution with degrees of freedom $f_1 = N-1, f_2 = N-1$.

Similarly, $u'/\gamma' = \chi_2'^2/\chi_1'^2$ has the F distribution with the same numbers of degrees of freedom (since $N = N'$, in the present case).

If the hypothesis H_1' is true (i.e., $\gamma = \gamma'$)

$$\frac{u}{u'} = \frac{\chi_2^2\chi_1'^2}{\chi_1^2\chi_2'^2} = \frac{\theta_1'\theta_2}{\theta_1\theta_2'} = \frac{z_1}{z_2}, \tag{19}$$

where $\theta_i(-\frac{1}{2}\chi_i^2)$ or θ_1' is distributed as

$$\frac{1}{\Gamma(a_i)}\theta_i^{a_i-1}e^{-\theta_i}\,d\theta_i, \tag{20}$$

with $a_i = \frac{1}{2}(N-1)$, and $z_1(=\theta_1'\theta_2)$, $z_2(=\theta_1\theta_2')$ follow independently the Wilks' z-distribution, [14], which we shall study in detail for the present case.

Distribution of z when $p = 2$: Consider

$$z = B\theta_1\theta_2\cdots\theta_p\,.$$

Wilks has succeeded in integrating the distribution of z for the case $p = 2$ for special values of a's, e.g., $a_1 = \frac{1}{2}(N-1)$, $a_2 = \frac{1}{2}(N-2)$. Now we want the distribution of z when $p = 2$ and for any values of a, and then for $a_1 = a_2 = \frac{1}{2}(N-1)$.

By (20) the joint distribution of θ_1 and θ_2 is

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)}\theta_1^{a_1-1}e^{-\theta_1}\theta_2^{a_2-1}e^{-\theta_2}\,d\theta_1\,d\theta_2.$$

Applying the transformation $z = B\theta_1\theta_2$, $v_1 = \theta_1$, the joint distribution of v_1, z is

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)}v_1^{a_1-1}e^{-v_1}\left(\frac{z}{Bv_1}\right)^{a_2-1}e^{-z/Bv_1}\frac{dv_1\,dz}{Bv_1}.$$

Integrating v_1 from $v_1 = 0$ to $v_1 = \infty$, we have the distribution of z, viz.,

$$\frac{z^{a_2-1}\,dz}{B^{a_2}\Gamma(a_1)\Gamma(a_2)}\int_0^\infty v^{a_1-a_2-1}e^{-v_1-z/Bv_1}\,dv_1. \tag{21}$$

In order to evaluate the integral of (20), consider the transformation $v_1 = y^2$, $dv_1 = 2y\,dy$, we have

$$I_0 = 2\int_0^\infty y^{2(a_1-a_2)-1}e^{-y^2-z/By^2}\,dy. \tag{22}$$

To evaluate I_0 for any a's, by putting $y = 1/x$, $dy = -dx/x^2$, we have

$$I_0 = 2\int_0^\infty \frac{e^{-zx^2/B-1/x^2}}{x^{2(a_1-a_2)+1}}\,dx. \tag{23}$$

Consider

$$\frac{\Gamma(a_1-a_2+\frac{1}{2})}{x^{2(a_1-a_2)+1}} = \int_0^\infty e^{-x^2y}y^{a_1-a_2-\frac{1}{2}}\,dy. \tag{24}$$

Then

$$I_0\Gamma(a_1 - a_2 + \tfrac{1}{2}) = 2\int_0^\infty e^{-(xz^2/B+1/z^2)}\,dx \int_0^\infty e^{-z^2y} y^{a_1-a_2-\frac{1}{2}}\,dy$$

$$= 2\int_0^\infty y^{a_1-a_2-\frac{1}{2}}\,dy \int_0^\infty e^{-[(z/B+y)x^2+1/x^2]}\,dx$$

$$= \sqrt{\pi}\int_0^\infty e^{-2\sqrt{z/B+y}}\, y^{a_1-a_2-\frac{1}{2}}\,\frac{dy}{\sqrt{z/B+y}}.$$

Since by the substitution $\sqrt{\frac{z}{B}+y} = \sqrt{\frac{z}{B}} + y$ or $y = x^2 + 2\sqrt{\frac{z}{B}}\,x$, $dy = 2\left(x + \sqrt{\frac{z}{B}}\right)dx$ and therefore

$$I_0\Gamma(a_1 - a_2 + \tfrac{1}{2}) = 2\sqrt{\pi}\int_0^\infty e^{-2(\sqrt{z/B}+x)}\left(x^2 + 2x\sqrt{\frac{z}{B}}\right)^{a_1-a_2-\frac{1}{2}}dx,$$

$$I_0 = \frac{\sqrt{\pi}\,e^{-2\sqrt{z/B}}}{\Gamma(a_1 - a_2 + \frac{1}{2})}\int_0^\infty e^{-2(\sqrt{z/B}+x)}\left(x^2 + 2x\sqrt{\frac{z}{B}}\right)^{a_1-a_2-\frac{1}{2}}dx. \tag{25}$$

Hence, z is distributed as

$$\frac{2\sqrt{\pi}\,z^{a_2-1}e^{-2\sqrt{z/B}}}{B^{a_2}\Gamma(a_1)\Gamma(a_2)\Gamma(a_1 - a_2 + \frac{1}{2})}\int_0^\infty e^{-2x}\left(2\sqrt{\frac{z}{B}} + x\right)^{a_1-a_2-\frac{1}{2}} x^{a_1-a_2-\frac{1}{2}}\,dx. \tag{26}$$

We infer from this distribution that when $2(a_1 - a_2)$, i.e., the difference of degrees of freedom, is odd, the integral can be expressed as a terminated series; but for even values of $2(a_1 - a_2)$, the series is infinite.

When $B = \frac{1}{A}$, $a_1 = \frac{1}{2}(N-1)$, $a_2 = \frac{1}{2}(N-2)$, (26) is reduced to

$$\frac{\sqrt{\pi}\,A^{a_2}z^{a_2-1}e^{-2\sqrt{Az}}}{\Gamma(a_1)\Gamma(a_2)}, \tag{27}$$

which is Wilks' ξ distribution, [15], for $p = 2$.

When $B = 1$ and $a_1 = a_2 = \frac{1}{2}(N-1)$, it becomes

$$\frac{2\sqrt{\pi}\,z^{a_2-1}e^{-2\sqrt{z}}}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty e^{-2x}(2\sqrt{z} + x)^{-\frac{1}{2}}x^{-\frac{1}{2}}\,dx, \tag{28}$$

which is the distribution of z involved in (19).

Since (28) can apparently not be simplified, I have been unable thus far to find in manageable form the distribution of the ratio z_1/z_2 and therefore of u/u' in this case. However, it would be simpler to use the alternative criterion $w = z - z'$ for the hypothesis H_1. The distribution of w will be taken up in a later section.

The distribution of v/v': Since Ns_Y^2/σ_Y^2 and $\Sigma X^2/\sigma_X^2$ have independently the χ^2 distribution with $N-1$ and N degrees of freedom respectively, therefore,

$$v = \frac{NS_Y^2}{\Sigma X^2} = \frac{\sigma_Y^2 \chi_2^2}{\sigma_X^2 \chi_1^2} = \frac{\gamma \chi_2^2}{\chi_1^2},$$

and $\frac{v}{\gamma} \Big/ \frac{N-1}{N}$ has the F-distribution with $f_1 = N-1$ degrees of freedom and $f_2 = N$.

Similarly $\frac{v'}{\gamma'} \Big/ \frac{N-1}{N}$ has the F-distribution with degrees of freedom f_1 and f_2 as above.

If the hypothesis H_2 is true (i.e., $\gamma = \gamma'$),

$$\frac{v}{v'} = \frac{\chi_2^2 \chi_1'^2}{\chi_1^2 \chi_2'^2} = \frac{\theta_1' \theta_2}{\theta_1 \theta_2'} = \frac{z_1}{z_2},$$

where each θ_i is distributed as in (19), but with $a_1 = \frac{1}{2}N$ and $a_2 = \frac{1}{2}(N-1)$. We can infer from (27) that $t_1 = 4\sqrt{z_1}$ and $t_2 = 4\sqrt{z_2}$ have independently the χ^2-distribution each with $4a_2$ or $2(N-1)$ degrees of freedom, and $t_1/t_2 = \sqrt{z_1/z_2} = \sqrt{v/v'}$ follows the F-distribution with degrees of freedom $f_1 = f_2 = 2(N-1)$. The 5% and 1% points of the $F = v/v'$ may be obtained from Snedecor's table ([12], p. 174).

5. **The distribution of $y = \log z$.** Wald [13] has suggested that the distribution of $z = B\theta_1\theta_2 \cdots \theta_p$ for any a_i's ($i = 1, \cdots, p$) may also be obtained indirectly with the aid of the characteristic function. A similar method has been applied in a recent paper by Wald and Brookner [14]. Consider the transformation

$$y = \log t = \log B\theta_1\theta_2 \cdots \theta_p. \tag{29}$$

The characteristic function of y is

$$\begin{aligned} \varphi_y(t) &= E(e^{ity}) = E\{(B\theta_1\theta_2 \cdots \theta_p)^t\} \\ &= \frac{B^t \Gamma(a_1+t)\Gamma(a_2+t) \cdots \Gamma(a_p+t)}{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_p)}. \end{aligned} \tag{30}$$

Thus the distribution $f(y)\,dy$ is given by

$$f(y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-ity} \varphi_y(t)\,dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} B^t e^{-ity} \prod_{i=1}^{p} \frac{\Gamma(a_i+t)}{\Gamma(a_i)}\,dt. \tag{31}$$

Without loss of generality, we may take $a_1 \geqq a_2 \geqq \cdots \geqq a_p > 0$ and let $a_p + t = -t'$, then

$$f(y) = \frac{c_p}{2\pi i} \int_{a_p - i\infty}^{-a_p + i\infty} e^{yt'} B^{-t'} \prod_{i=1}^{p} \Gamma(a_i - a_p - t')\,dt', \tag{32}$$

where $c_p = e^{a_p y} B^{-a_p} \Big/ \prod_{i=1}^{p} \Gamma(a_i)$.

The integration can be carried out by the method of residue along the contour C, bounded by the line $x = -a_p$ and that part of the circle with center at origin and radius r, which lies to the right of the line $x = -a_p$. The integral of the function $e^{t'y}B^{-t'}\prod_{i=1}^{p}\Gamma(a_i - a_p - t')$ along the arc converges to zero as the radius of the circle tends to infinity (Kullback, [8]). Hence the integrals along the vertical line $x + a_p = 0$ and along the closed contour C are equal. Then we may write

$$f(y) = -\frac{c_p}{2\pi i}\int_C e^{yt'}B^{-t'}\prod_{i=1}^{p}\Gamma(a_i - a_p - t')\,dt', \tag{33}$$

and its value is c_p times the sum of the residues at the poles within the contour C.

For the present purpose, $p = 2$, we have

$$f(y) = \frac{c_2}{2\pi i}\int_{-a_2+i\infty}^{-a_2+i\infty} e^{yt'}\Gamma(a_1 - a_2 - t')\Gamma(-t')\,dt'. \tag{34}$$

We shall study the integral of (34) in more detail in the following cases:

(i) $a_1 - a_2 = \frac{1}{2}$. By the duplication formula

$$\Gamma(\tfrac{1}{2} - t')\Gamma(-t') = 2^{1+2t'}\sqrt{\pi}\,\Gamma(-2t'),$$

and the function

$$\Gamma(-2t') = \lim_{N\to\infty}\frac{N!N^{-2t'}}{(-2t')(-2t'+1)\cdots(-2t'+N)},$$

has simple poles at the points $0, \frac{1}{2}, 1, 3/2, \cdots$. The residue at $t' = m/2$, where m is zero or a positive integer, is $(-1)^{m+1}/2\cdot m!$ and (34) becomes

$$\begin{aligned} f(y) &= \sqrt{\pi}\,c_2\left(1 - 2e^{\frac{1}{2}y} + \frac{1}{2!}2^2e^{y} - \frac{1}{3!}2^3e^{3y/2} + \cdots\right) \\ &= \sqrt{\pi}\,c_2e^{-2e^{\frac{1}{2}y}}. \end{aligned} \tag{35}$$

The distribution of $z = e^y$ is

$$\frac{2\sqrt{\pi}\,z^{a_2-1}e^{-2\sqrt{z}}}{\Gamma(a_1)\Gamma(a_2)}\,dz. \tag{27 bis}$$

(ii) $a_1 - a_2 = m + \frac{1}{2}$. The function

$$\begin{aligned} \Gamma(a_1 - a_2 - t')\Gamma(-t') &= (m - \tfrac{1}{2} - t')(m - \tfrac{3}{2} - t')\cdots(\tfrac{1}{2} - t')\Gamma(\tfrac{1}{2} - t')\Gamma(-t') \\ &= 2^{1+2t'}\sqrt{\pi}(m - \tfrac{1}{2} - t')(m - \tfrac{3}{2} - t')\cdots(\tfrac{1}{2} - t')\Gamma(-2t') \end{aligned}$$

has simple poles at 0, m, $m + \frac{1}{2}$, $m + 1$, $\cdots$, and

$$f(y) = \sqrt{\pi}\, c_2 \left[\frac{(2m-1)!}{2^{2m-1}(m-1)!} - \frac{1}{2^m(2m)} (2^2 e^y)^m + \frac{1}{2^m(2m+1)} (2^2 e^y)^{m+1} \right.$$
$$\left. - \frac{1}{2^m(2m+2)\cdot 2\cdot 1} (2^2 e^y)^{m+1} + \cdots \right]$$
$$= \sqrt{\pi}\, c_2 \left[\frac{(2m-1)!}{2^{2m-1}(m-1)!} - \frac{1}{2^m} \sum_{\gamma=0}^{\infty} \frac{1}{(2m+\gamma)\gamma!} (2^2 e^y)^{m+\gamma/2} \right].$$

This agrees with the expansion of (26) when we put $a_1 - a_2 - \frac{1}{2} = m$.

(iii) $a_1 - a_2 = 0$. The function

$$[\Gamma(-t')]^2 = \lim_{N\to\infty} \frac{(N!)^2 N^{-2t'}}{(-t')^2(-t'+1)^2 \cdots (-t'+N)^2},$$

has poles of the second order at the points 0, 1, 2, 3, $\cdots$ and

$$f(y) = c_2 \sum_{\gamma=0}^{\infty} \frac{d}{dt'} \{(t'-\gamma)^2 e^{t'y} [\Gamma(-t')]^2\}_{t'=\gamma}$$

(iv) $a_1 - a_2 = m$. The function

$$\Gamma(m - t')\Gamma(-t') = (m-1-t')(m-2-t') \cdots (1-t')(-t')[\Gamma(-t')]^2,$$

has finite simple poles at 1, 2, $\cdots$, $m-1$ and poles of the second order at m, $m+1$, $\cdots$, and

$$f(y) = c_2 \sum_{\gamma=0}^{m-1} \{(t-\gamma)e^{t'y}\Gamma(m-t')\Gamma(-t')\}_{t'=\gamma}$$
$$+ c_2 \sum_{\gamma=m}^{\infty} \left\{ \frac{d}{dt'} (t'-\gamma)^2 e^{t'y} \Gamma(m-t')\Gamma(-t') \right\}_{t'=\gamma}.$$

6. **The distribution of $w = z - z'$ or $\psi = \cosh w$.** Since the distribution of u is given in [7] as

$$\frac{1}{\gamma B[\frac{1}{2}(N-1), \frac{1}{2}(N-1)]} \left(\frac{u}{\gamma}\right)^{\frac{1}{2}N-3} \left(1 + \frac{u}{\gamma}\right)^{-(N-1)} du, \tag{39}$$

therefore, by transformation (17), we have that the distribution of z for a given $\zeta = \frac{1}{2}\log\gamma = \frac{1}{2}\log\frac{1+\rho}{1-\rho}$ is

$$\frac{1}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \operatorname{sech}^n (z - \zeta)\, dz, \tag{40}$$

where $n = N - 1$. The distribution of z has been given by R. A. Fisher [3] for $n = 1$ and by Delury [2]. Similarly, the distribution of z' for a given ζ' is

$$\frac{1}{\mathrm{B}\left(\frac{1}{2}, \frac{n'}{2}\right)} \operatorname{sech}^{n'}(z' - \zeta')\, dz', \tag{41}$$

where $n' = N' - 1$.

In case $n = n'$, the joint distribution of z and z' for a given common ζ is

$$C \operatorname{sech}^{n}(z - \zeta) \operatorname{sech}^{n}(z' - \zeta)\, dz\, dz' = \frac{C\, dz\, dz'}{\cosh^{n}(z - \zeta) \cosh^{n}(z' - \zeta)}, \tag{42}$$

where $1/C = \left[\mathrm{B}\left(\frac{1}{2}, \frac{n}{2}\right)\right]^2$.

By the transformation $\bar{z} = \frac{1}{2}(z + z')$, $w = z - z'$, we have the joint distribution of $\bar{z}$ and w,

$$\frac{C\, d\bar{z}\, dw}{[\cosh^{n}(z - \zeta) \cosh^{n}(z' - \zeta)]} = \frac{2^{n} C\, d\bar{z}\, dw}{[\cosh 2(\bar{z} - \zeta) + \cosh w]^{n}}. \tag{43}$$

Integrating with respect to $\bar{z}$ from $-\infty$ to ∞, we have

$$\begin{aligned} 2^{n} C\, dw \int_{-\infty}^{\infty} \frac{d\bar{z}}{[\cosh 2(\bar{z} - \zeta) + \cosh w]^{n}} & \\ = 2^{n} C\, dw \int_{0}^{\infty} \frac{2\, d\bar{z}}{[\cosh 2(\bar{z} - \zeta) + \cosh w]^{n}} & \\ = 2^{n} C\, dw I_{n}, \text{ say.} & \end{aligned} \tag{44}$$

Applying the transformation $\phi = 2(\bar{z} - \zeta)$, $\psi = \cosh w$, the integral of (34) becomes

$$I_{n} = \int_{0}^{\infty} \frac{d\phi}{(\cosh \phi + \psi)^{n}}.$$

Substituting $\cosh \phi + \psi = \dfrac{1 + \psi}{\theta}$, we have

$$\begin{aligned} I_{n} &= \int_{0}^{1} \left(\frac{\theta}{1 + \psi}\right)^{n} \frac{1}{\theta} \frac{d\theta}{\sqrt{\left(1 - \dfrac{\psi - 1}{\psi + 1}\theta\right)(1 - \theta)}} \\ &= \frac{1}{(\psi + 1)^{n}} \int_{0}^{1} \theta^{n - 1}(1 - \theta)^{-\frac{1}{2}} \left(1 - \frac{\psi - 1}{\psi + 1}\theta\right)^{-\frac{1}{2}} d\theta. \end{aligned} \tag{45}$$

Comparing (35) with the hypergeometric function

$$I = \int_{0}^{1} \theta^{b - 1}(1 - \theta)^{c - b - 1}(1 - \theta x)^{-a}\, d\theta = \frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)} F(a, b, c, x), \tag{46}$$

we have $b = n$, $c - b = \frac{1}{2}$, $a = \frac{1}{2}$, and therefore (35) can be expressed in terms of a hypergeometric series as

$$I_n = \frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \frac{1}{(\psi+1)^n} F\left(\tfrac{1}{2}, n, n+\tfrac{1}{2}, \frac{\psi-1}{\psi+1}\right). \tag{47}$$

The series (37) is convergent since $\frac{\psi-1}{\psi+1}$ is less than unity. Thus the distribution of w, from (34), is

$$\frac{2^n C\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \frac{1}{(\cosh w+1)^n} F\left(\tfrac{1}{2}, n, n+\tfrac{1}{2}, \frac{\cosh w-1}{\cosh w+1}\right) dw, \tag{48}$$

and the distribution of $\psi = \cosh w$ is

$$\frac{2^{n+1} C\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} \frac{1}{(\psi+1)^{n+\frac{1}{2}}(\psi-1)^{\frac{1}{2}}} F\left(\tfrac{1}{2}, n, n+\tfrac{1}{2}, \frac{\psi-1}{\psi+1}\right) d\psi. \tag{49}$$

We notice that the distribution of ψ expressed in (39) is very similar to the r-distribution expressed in terms of hypergeometric series, except that in the first case the argument is $\frac{\psi-1}{\psi+1}$, while in the second case it is $\frac{1-p}{1+p}$ where $p = \rho r$. Hotelling [5] has obtained a very rapidly convergent hypergeometric series for the distribution of the correlation coefficient since $|p| < 1$. But for the distribution of ψ, we cannot obtain a more rapidly convergent series than (39), since the values of ψ lie between 1 and ∞.

7. Summary and remark. Two hypotheses concerning the comparison of correlation coefficients of two samples from bivariate normal populations have been considered. The appropriate test criteria for each hypothesis have been derived by the use of a transformation of the variates. The distributions of certain of the criteria have been obtained in the special case where $N = N'$. Incidentally the distribution of Wilks' z for $p = 2$ and any values of a_1 and a_2 has been derived.

Again though we assume throughout the paper that $\sigma_1 = \sigma_2$ and $\sigma_1' = \sigma_2'$, the tests can be generalized to fit the case where the ratios $\sigma_1/\sigma_2 = k$, $\sigma_1'/\sigma_2' = k'$ are known, but are different from unity. In the latter case we can apply the transformation

$$y_1 = w_1 x_1, \qquad y_2 = w_2 x_2;$$
$$y_1' = w_1' x_1', \qquad y_2' = w_2' x_2';$$

where

$$w_1 k_1 = w_2 k_2 = 1, \qquad w_1' k_1' = w_2' k_2' = 1,$$

so that after transformation the variances of each pair of y's are equal.

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REFERENCES

[1] F. A. BRANDER, *Biometrika*, Vol. 35 (1933), p. 102.
[2] D. B. DELURY, *Annals of Math. Stat.*, Vol. 9 (1938), p. 145.
[3] R. A. FISHER, *Metron*, Vol. 1 (1921), p. 3.
[4] H. HOTELLING, *Biometrika*, Vol. 26 (1936), p. 321.
[5] H. HOTELLING, Lectures delivered at Columbia University (1940–41).
[6] H. HOTELLING, *Annals of Math. Stat.*, Vol. 11 (1940), p. 271.
[7] C. T. HSU, *Annals of Math. Stat.*, Vol. 11 (1940), p. 410.
[8] S. KULLBACK, *Annals of Math. Stat.*, Vol. 5 (1934), p. 263.
[9] J. NEYMAN AND E. S. PEARSON, *Biometrika*, Vol. 20 (1928), p. 175.
[10] J. NEYMAN AND E. S. PEARSON, *Bull. Acad. Polonaise Sci. Lettres A* (1931), p. 460.
[11] E. S. PEARSON AND J. NEYMAN, *Bull. Acad. Polonaise Sci. Lettres A* (1930), p. 73.
[12] G. W. SNEDECOR, *Statistical Methods*, Collegiate Press, Ames, Iowa, 1937.
[13] A. WALD, Lectures delivered at Columbia University (1939–40).
[14] A. WALD AND R. J. BROOKNER, *Annals of Math. Stat.*, Vol. 12 (1941), p. 137.
[15] S. S. WILKS, *Biometrika*, Vol. 24 (1932), p. 471.

ON RANDOMNESS IN ORDERED SEQUENCES

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It is frequently desirable to examine an ordered sequence of measurements for the presence of non-random variability, concern over any particular type of variability being limited. Unless the sequence is one containing replicated observations, current methods of analysis often restrict an investigation to tests for specific forms of variability, such as particular orders of regression and periodicity. In order to simulate replication, arbitrary grouping of data is occasionally used and followed by some test of variance; this practice, however, is likely to add an element of bias to the investigation.

Under these conditions, it would be convenient to have the means of testing a series for the presence of general regression, before proceeding to test for that of a specific type. It is the purpose of this paper to present, as briefly as possible, a statistic designed for this preliminary type of examination, and to demonstrate its application.

If a given sequence of measurements be denoted by

$$X_1, X_2, \cdots, X_n$$

then the magnitude of

$$C = 1 - \frac{\sum_1^{n-1} (X_i - X_{i+1})^2}{2 \sum_1^n (X_i - \bar{X})^2},$$

will be dependent upon the arrangement of the n observations upon which it is based. C will have $n!$ possible values for a given sample, corresponding to the number of permutations of n items.

1. Moments of the distribution of C in terms of the moments of a finite sequence. Writing C in terms $x_1, \cdots, x_n$, representing the deviations of $X_1, \cdots, X_n$ from their sample mean of n measurements,

$$C = 1 - \frac{\sum_1^{n-1} (x_i - x_{i+1})^2}{2 \sum_1^n x_i^2}$$

$$= \frac{x_1^2 + x_n^2 + 2 \sum_1^{n-1} x_i x_{i+1}}{2 \sum_1^n x_i^2}.$$

In order to find the mean value of C for a given sample, it must be summed over all values obtained from the $n!$ permutations of the measurements.

Dealing with the numerator alone of the expression given above:

$$\sum_p \left[x_1^2 + x_n^2 + 2 \sum_1^{n-1} x_i x_{i+1} \right] = \sum_p x_1^2 + \sum_p x_n^2 + 2 \sum_p \sum_1^{n-1} x_i x_{i+1},$$

where Σ_p denotes summation over the $n!$ permutations.

There are n values of x_i, and $n!$ arrangements. Each value x_i is x_1 in $(n-1)!$ of the arrangements: the same reasoning applies to x_n. The first two terms of the summation, therefore, will be

$$\sum_p x_1^2 = \sum_p x_n^2 = (n-1)! \sum_1^n x_i^2.$$

With regard to the third term, there are $2(n-1)$ of such cross-products for each arrangement. Since the summation is taken over $n!$ arrangements, $x_j x_k$ will be different than $x_k x_j$, and should be considered a separate term. Each crossproduct term, therefore, must occur $\dfrac{2(n!)(n-1)}{n(n-1)}$ times throughout the $n!$ arrangements, since there are $n(n-1)$ possible cross-products among n different items. The third term, then, will be

$$2 \sum_p \left(\sum_1^{n-1} x_i x_{i+1} \right) = 2(n-1)! \sum_1^n \sum_1^{n-1} x_j x_k = -2(n-1)! \sum_1^n x_i^2,$$

from which it may be seen that the mean value of C is zero for any sample.

The same method may be applied in order to find the second and higher moments of C. Squaring the numerator of the expression and expanding,

$$\sum_p \left[x_1^2 + x_n^2 + 2 \sum_1^{n-1} x_i x_{i+1} \right]^2$$

$$= \sum_p \left[x_1^4 + x_n^4 + 2x_1^2 x_n^2 + 4x_1^2 \sum_1^{n-1} x_i x_{i+1} + 4x_n^2 \sum_1^{n-1} x_i x_{i+1} + 4 \left(\sum_1^{n-1} x_i x_{i+1} \right)^2 \right].$$

Performing the summation Σ_p term by term we obtain

$$\frac{\sum_p \left[x_1^2 + x_n^2 + 2 \sum_1^{n-1} x_i x_{i+1} \right]^2}{n!} = \frac{2(2n-3) \left(\sum_1^n x_i^2 \right)^2 - 2n \sum_1^n x_i^4}{n(n-1)},$$

whence the second moment of C for any sample is given by

$$M_2 = \frac{2n - 3 - m_4/m_2^2}{2n(n-1)},$$

where m_2 and m_4 are the second and fourth moments, respectively, of the n observations about their mean.

In like manner, the third and fourth moments of the distribution of C for a given sample of n observations are found to be

$$M_3 = \frac{-6 + 4(n-3)\dfrac{m_3^2}{m_2^3} + 9\dfrac{m_4}{m_2^2} - 3\dfrac{m_6}{m_2^3}}{4n(n-1)(n-2)},$$

$$M_4 = \frac{1}{8n^3(n-1)(n-2)(n-3)}\left[24n^2(n-3)^2 - 48n(4n-9)\frac{m_3^2}{m_2^3}\right.$$

$$- 24n(3n^2 - 17n + 27)\frac{m_4}{m_2^2} + (8n^3 - 45n^2 - 23n + 210)\frac{m_4^2}{m_2^4}$$

$$+ 16(2n^2 + 5n - 21)\frac{m_5 m_3}{m_2^4} + 4(17n^2 - 37n + 42)\frac{m_6}{m_2^3}$$

$$\left. - (7n^2 + 13n - 6)\frac{m_8}{m_2^4}\right].$$

2. Distribution of C for samples drawn from a normal universe. The first four moments of the distribution of C for samples drawn from a given population may be derived from the above formulae by substituting the mean values of $\dfrac{m_3^2}{m_2^3}$, $\dfrac{m_4}{m_2^2}$, etc. of samples from such a population. For normal samples containing n observations, for example, the following mean values apply, as obtained by the method presented by R. A. Fisher [1, 2]:

$$\frac{m_3^2}{m_2^3} = \frac{6(n-2)}{(n+1)(n+3)},$$

$$\frac{m_4}{m_2^2} = \frac{3(n-1)}{(n+1)},$$

$$\frac{m_4^2}{m_2^4} = \frac{3(3n^3 + 23n^2 - 63n + 45)}{(n+1)(n+3)(n+5)},$$

$$\frac{m_5 m_3}{m_2^4} = \frac{60(n-1)(n-2)}{(n+1)(n+3)(n+5)},$$

$$\frac{m_6}{m_2^3} = \frac{15(n-1)^2}{(n+1)(n+3)},$$

$$\frac{m_8}{m_2^4} = \frac{105(n-1)^3}{(n+1)(n+3)(n+5)}.$$

Replacement of the sample moment ratios by the mean values of those ratios for normal samples yields the following moments of C:

$$M_1 = 0, \qquad M_2 = \frac{n-2}{(n-1)(n+1)}, \qquad M_3 = 0,$$

$$M_4 = \frac{3(n^2 + 2n - 12)}{(n-1)(n+1)(n+3)(n+5)}.$$

Compatible results for the case of normal samples have been obtained by Williams [3], using another method.

From the above results, the value of

$$\beta_2 = \frac{3(n^2 + 2n - 12)(n - 1)(n + 1)}{(n - 2)^2(n + 3)(n + 5)},$$

is seen to approach normality as the sample size is increased.

Inasmuch as the distribution of C for normal samples is limited in both directions and is symmetrical, it is apparent that the Pearson Type II distribution may be considered representative. Fitting this curve to the moments given above, the equation of the frequency distribution is given by

$$y = y_0\left(1 - \frac{C^2}{a^2}\right)^m,$$

where

$$m = \frac{(n^4 - n^3 - 13n^2 + 37n - 60)}{2(n^3 - 13n + 24)},$$

$$a^2 = \frac{(n^2 + 2n - 12)(n - 2)}{(n^3 - 13n + 24)},$$

$$y_0 = \frac{\Gamma(2m + 2)}{a \cdot 2^{2m+1}[\Gamma(m + 1)]^2}.$$

The values of β_2 for the distribution, for various values of n, are as follows:

Sample size, n	β_2
5	2.300
10	2.570
15	2.684
20	2.750
25	2.793
50	2.833

Due to the effect of even moments higher than the fourth, the approximation afforded by the Type II curve is not reliable for samples containing less than about eight observations. As the sample size decreases below this limit, the extremes of the C distribution deviate increasingly from the extremes ($\pm a$) of the fitted curve: with such a platykurtic distribution, therefore, the effect upon the lower significance levels vitiates the approximation.

Although either β_2 or the theoretical limits of the distribution of C could have been employed as a parameter of the fitted curve, it was considered expedient to use the former. In any case, of course, the advantage to be gained would be in connection only with samples containing few observations (less than eight). The evidence afforded by empirical sampling indicates that use of the limits as a parameter might render the approximation less valid.

In order to facilitate use of the approximate distribution for samples of eight or more observations, the values of C associated with two probability levels are tabulated below in Table I. The ratio of each value of C to its standard error is also shown, to demonstrate the approach to normality. The significance levels recorded exclude 10% and 2% of the area under the curve, respectively. In most practical applications, these will be the 5% and 1% levels, respectively, since only positive values of C exceeding the tabulated value will ordinarily be considered significant. The tabulations were prepared from tables of the function $I_x(p, q)$ [5], where $q = .5$ and $p = m + 1$, with the transformation $x = 1 - \frac{C^2}{a^2}$.

TABLE I

Significance levels of the absolute value of C

Sample size, n	$P = .10$	$C_{.10}/\sigma_c$	$P = .02$	$C_{.02}/\sigma_c$
8	.5088	1.6486	.6686	2.1664
9	.4878	1.6492	.6456	2.1826
10	.4689	1.6494	.6242	2.1958
11	.4517	1.6495	.6044	2.2068
12	.4362	1.6495	.5860	2.2161
13	.4221	1.6495	.5691	2.2241
14	.4092	1.6494	.5534	2.2310
15	.3973	1.6493	.5389	2.2369
16	.3864	1.6492	.5254	2.2423
17	.3764	1.6492	.5128	2.2470
18	.3670	1.6491	.5011	2.2513
19	.3583	1.6489	.4900	2.2550
20	.3502	1.6488	.4797	2.2585
21	.3426	1.6488	.4700	2.2616
22	.3355	1.6486	.4609	2.2647
23	.3288	1.6485	.4521	2.2676
24	.3224	1.6484	.4440	2.2700
25	.3165	1.6484	.4361	2.2717
Normal ($n = \infty$)		1.6447		2.3262

The distribution of C for normal samples containing 20 or more observations is sufficiently normal, for most practical cases and for the more common significance levels, to permit use of a table of areas under the normal curve, in conjunction with the standard error $\sigma_c = \sqrt{\frac{n-2}{(n-1)(n+1)}}$. The 5% significance levels shown in Table I result, at worst, in a one per cent error of probability estimate, if the normal approximation is used in their place: that is, if 1.6447 times the standard error is used instead of the tabulated significance level, the probability will be .0505 at most, for the values of n which are tabulated.

3. General discussion on the application of C. It may be wondered why the statistic C has been used, rather than the more easily computed statistic $C' = \dfrac{\sum_1^{n-1} (X_i - X_{i+1})^2}{\sum_1^n x_i^2}$. As far as a significance test is concerned, it clearly does not matter which is used, since C and C' are linearly related. However, C may be regarded as symmetrically distributed about 0 in samples from a normal population to within at least four moments. Excessive departure of C from 0 may be taken as indicative of the presence of non-randomness in the series, the actual significance test being based, of course, on the probability of obtaining a departure larger than a given observed one, under the assumption of a random series. Positive values of C, in general, correspond to positive correlation while negative values correspond to negative correlation between successive observations.

There are various ways of detecting non-randomness in a series of observations, such as regression methods, analysis of variance, etc. The use of regression methods implies that we must know in general the type of regression function to be tried. C is a very flexible statistic, on the other hand, for testing the null hypothesis that a series is random, no matter what the alternative hypothesis is. A thorough study of C as a statistic for testing the hypothesis of randomness in an ordered series should include a study of the power function of C for hypotheses specifying various types of non-randomness. However, we shall simply appeal to intuition in proposing the statistic C, and forego power function considerations in this note. In practice, the advantage of using C increases with the length of a series: lack of randomness in a single sequence of ten or less observations may ordinarily be detected by regression methods, in fitting a low order polynomial. In a longer sequence of measurements, on the other hand, the presence of complicated regression or of periodicity is often sufficiently obscured by variation to elude detection by any other than a flexible method.

The statistic could be used to advantage in the field of applied statistics, in the investigation not only of variate series but of attribute series as well. For the latter purpose, an effort to tabulate the relationship between the level of significance and the percentage of either attribute would facilitate statistical investigation of random arrangement. A direct application could thus be made to binomially distributed attributes by a scalar assignment (0, 1) to the dichotomy, followed by a procedure similar to that presented above. Similarly, the randomness of vectorial observations could be examined from the viewpoint of arrangement. The common method of treating such problems,—the "random walk method,"—has occasionally been found inadequate in dealing with specific forms of non-random order; this is especially true when the allocable cause of variation has a multi-directional effect.

Needless to say, each of the fields of application considered so briefly above would require development before a routine, efficient method of investigating ordered arrangement could be established. Although probability level tables

have been provided in this paper for C as applied to normal samples, it is quite evident that tables for samples from other parent distributions would be needed for some of the applications mentioned above.

4. An illustration of the use of C. Although one example has already been presented elsewhere [4] in which the distribution developed in Section 2 has been employed, a typical application of the statistic to an example in the field of quality control will be given here in order to illustrate the mechanics of solution. The data presented in Table II represent the percentages of defective product turned out daily, over a period of twenty-four days, by a single workman. The total output each day closely approximates five hundred parts: this fact is brought out to explain the calculation of χ^2 for the observed series of percentages, —it has no bearing upon the use of C.

TABLE II

Percentage of product rejected

Day	%, X	X^2	d^2
1	7.4	54.76	
2	8.8	77.44	1.96
3	11.4	129.96	6.76
4	10.3	106.09	1.21
5	11.9	141.61	2.56
6	12.2	148.84	.09
7	10.0	100.00	4.84
8	8.4	70.56	2.56
9	9.4	88.36	1.00
10	10.9	118.81	2.25
11	9.9	98.01	1.00
12	11.8	139.24	3.61
13	10.0	100.00	3.24
14	8.9	79.21	1.21
15	9.7	94.09	.64
16	9.3	86.49	.16
17	12.0	144.00	7.29
18	12.3	151.29	.09
19	10.3	106.09	4.00
20	8.6	73.96	2.89
21	10.4	108.16	3.24
22	11.1	123.21	.49
23	9.4	88.38	2.89
24	8.2	67.24	1.44
Totals	242.6	2495.82	55.42
		$n\bar{X}^2$ 2452.28	
		$\Sigma x^2 =$ 43.54	

$C = .3636$ (significant) $\chi^2 = 21.518$ (23 degrees of freedom) (not significant).

The value of C derived from the data lies between the two significance levels tabulated in Table I; there is reason to believe that the data are ordered, or non-random. Computation of χ^2, however, has been carried out with the hypothesis that all product was made under the same conditions (i.e. with a percentage defective equal to 10.108%, the mean of the group). The value so obtained is associated with a probability of about $P = .50$: the hypothesis is not disproved by this test. In short, the variability of the twenty-four observations could be considered random if it were not for the order of their arrangement.

REFERENCES

[1] R. A. FISHER, "Moments and product moments of sampling distributions," *Lond. Math. Soc. Proc.* (series 2) 30 (1929), pp. 199–238.

[2] R. A. FISHER, "The moments of the distribution for normal samples of measures of departure from normality," *Roy. Soc. Proc.*, A 130 (1930), pp. 16–28.

[3] J. D. WILLIAMS, "Moments of the ratio of the mean square successive difference in samples from a normal universe," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 239–241.

[4] L. C. YOUNG, "A critical appraisal of statistical methods in industrial management," presented at the annual meeting, American Society of Mechanical Engineers (1940).

[5] K. PEARSON (Editor), *Tables of the Incomplete Beta-Function*, Biometrika Office, London, 1924.

ON CERTAIN LIKELIHOOD-RATIO TESTS ASSOCIATED WITH THE EXPONENTIAL DISTRIBUTION

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Various likelihood-ratio tests and their distributions in samples from a population having the elementary probability law $\frac{1}{\sigma}e^{-(x-B)/\sigma}$, $B \leq x \leq \infty$, have been studied by Neyman and Pearson [1] and Sukhatme [2]. In this note the power functions and the question of bias of several likelihood-ratio tests will be investigated. The exponential distribution appears to be appropriate for dealing with problems involving the intervals of time between events which tend to be random, as for example the interval between consecutive telephone calls, or the interval between consecutive accidents to the same worker.

To test the hypothesis H' that the location parameter B is equal to some fixed value, it being assumed that the scale parameter σ is known, we can for simplicity take the set Ω of admissible populations from which the sample might have been drawn to be $\{-\infty < B < +\infty, \sigma = 1\}$, while the subset ω from which the sample must come when the hypothesis is true is $\{B = 0, \sigma = 1\}$. Then the likelihood-ratio λ_1 for testing this hypothesis is

$$\lambda_1 = \frac{P(\omega \text{ max.})}{P(\Omega \text{ max.})} = \frac{e^{-\sum_{i=1}^{n} x_i}}{e^{-\sum_{i=1}^{n}(x_i - x_1)}} = e^{-nx_1},$$

where x_1 is the smallest observation in a random sample of n. The region of acceptance of this hypothesis consists of all points in sample space for which

$$\lambda_{1\epsilon} \leq \lambda_1 \leq 1,$$

where $\lambda_{1\epsilon}$ is chosen so that $\int_{\lambda_{1\epsilon}}^{1} g_1(\lambda_1)\, d\lambda_1 = 1 - \alpha$, α being the level of significance used and $g(\lambda_1)\, d\lambda_1$ being the distribution of λ_1 when B is really equal to zero. The region $\lambda_{1\epsilon} \leq \lambda_1 \leq 1$ is equivalent to the region in the sample space for which

$$0 \leq x_1 \leq k_1 \,;\, k_1 = -\frac{\log \lambda_{1\epsilon}}{n}.$$

For any value of B the distribution of x_1 is known [3] to be

$$\phi_1(x_1)\, dx_1 = ne^{-n(x_1 - B)}\, dx_1 .$$

Setting $B = 0$, the relationship between k_1 and α is

$$\int_0^{k_1} ne^{-nx_1}\,dx_1 = 1 - \alpha, \quad \text{so} \quad e^{-nk_1} = \alpha.$$

When $B \leq 0$, the power function $P(B)$, for this test is

$$P(B) = 1 - \int_0^{k_1} ne^{-n(x_1 - B)}\,dx_1 = 1 - e^{nB}[1 - \alpha].$$

When $0 \leq B \leq k_1$, $P(B) = 1 - \int_B^{k_1} ne^{-n(x_1 - B)}\,dx_1 = \alpha e^{nB}$. When $B \geq k_1$, $P(B) = 1$.

Since $e^{nB} > 1$ if $B > 0$ and also $e^{nB} < 1$ if $B < 0$, $P(B)$ is obviously $> \alpha$ if $B \neq 0$. This test is therefore completely unbiased in the sense of Daly [4]. In addition, it is not difficult to prove that this test has the unusual property of being a uniformly most powerful test with respect to all alternatives.

To test the hypothesis H'' that the location parameter is equal to some fixed value, say $B = 0$, when the scale parameter σ is unknown, the likelihood-ratio is easily seen to be

$$\lambda_2 = \left[\frac{\sum_{i=1}^{n}(x_i - x_1)}{\sum_{i=1}^{n} x_i}\right]^n = \left[\frac{1}{1 + \dfrac{nx_1}{\sum_{i=1}^{n}(x_i - x_1)}}\right]^n.$$

The region of acceptance consists of all points in the sample space for which $\lambda_{2\epsilon} \leq \lambda_2 \leq 1$ where $\int_{\lambda_{2\epsilon}}^{1} g_2(\lambda_2)\,d\lambda_2 = 1 - \alpha$. This is equivalent to the region

$$(1) \qquad 0 \leq \left[\frac{n(n-1)x_1}{\sum_{i=1}^{n}(x_i - x_1)} = t\right] \leq k_2\,; \qquad k_2 = (n-1)\,\frac{(1 - \lambda_{2\epsilon}^{1/n})}{\lambda_{2\epsilon}^{1/n}}.$$

The relation between k_2 and α is easily found from the distribution of t when $B = 0$, which is known to be [3]

$$\phi_2(t)\,dt = \frac{dt}{\left[1 + \dfrac{t}{n-1}\right]^n}.$$

Therefore $\int_0^{k_2} \phi_2(t)\,dt = 1 - \alpha$, so $\left[1 + \dfrac{k_2}{n-1}\right]^{-(n-1)} = \alpha$.

It is somewhat easier to find the power function of this test by considering the region of acceptance as made up of points in the x_1, s plane for which

$$0 \leq x_1 \leq \frac{k_2 s}{n} \quad \text{where} \quad s = \frac{\sum_{i=1}^{n}(x_i - x_1)}{n-1},$$

which is identical with the region in (1).

The joint distribution of x_1 and s is [3]

$$\psi_1(x_1, s)\, dx_1\, ds = \phi_3(x_1)\, dx_1 \cdot \phi_4(s)\, ds,$$

where

$$\phi_3(x_1)\, dx_1 = \frac{n}{\sigma} e^{-n(x_1 - B)/\sigma}\, dx_1$$

and

$$\phi_4(s)\, ds = \frac{\left(\frac{n-1}{\sigma}\right)^{n-1} s^{n-2} e^{-(n-1)s/\sigma}\, ds}{(n-2)!}.$$

When $B \leq 0$, the power function $P(B)$ of this test is

$$P(B) = 1 - \int_0^\infty ds \int_0^{k_2 s/n} \psi_1(x_1 s)\, dx_1 = 1 - e^{nB/\sigma}[1 - \alpha].$$

When $B \geq 0$, the power function is

$$\begin{aligned} P(B) &= 1 - \int_{Bn/k_2}^\infty ds \int_B^{k_2 s/n} \psi_1(x_1, s)\, dx_1 \\ &= \alpha e^{nB/\sigma} + I\left[n-1; \frac{n(n-1)B}{\sigma k_2}\right] - \alpha e^{nB/\sigma} I\left[n-1; \frac{n(n-1+k_2)B}{\sigma k_2}\right], \end{aligned} \tag{2}$$

where $I[p; x] = \dfrac{\Gamma_x(p)}{\Gamma(p)} = \dfrac{\int_0^x x^{p-1} e^{-x}\, dx}{\int_0^\infty x^{p-1} e^{-x}\, dx}$,

which is the form in which the Incomplete Gamma Function has been tabulated [5].

Since σ must be positive, $e^{nB/\sigma} < 1$ if $B < 0$ and therefore $P(B) > \alpha$ in the interval $-\infty < B < 0$. To show that $P(B)$ is $> \alpha$ in the interval $0 < B < \infty$, it is simpler to work with the expression for $P(B)$ as a double integral in (2), than to differentiate the power function directly. Performing the integration with respect to x_1,

$$P(B) = 1 + \int_{Bn/k_2}^\infty [e^{-(k_2 s - Bn)/\sigma} - 1] \cdot \phi_4(s)\, ds.$$

Differentiating with respect to B,

$$P'(B) = \int_{Bn/k_2}^\infty \frac{n}{\sigma} e^{-(k_2 s - Bn)/\sigma} \phi_4(s)\, ds.$$

The integral expression for $P'(B)$ is obviously positive. Therefore since for $B > 0$ the derivative is always positive the function must be monotomically

increasing in this interval ($0 < B < +\infty$), so $P(B)$ is $> \alpha$ when $B > 0$. Therefore this test is also completely unbiased.

We now consider the hypothesis H''' that two samples are drawn from exponential distributions with the same location parameter, assuming it is known the samples must have come from two exponential distributions with the same scale parameter. Given a sample of n_1 values of x drawn from $\frac{1}{\sigma} e^{-(x-B_1)/\sigma}\, dx$ and another independent sample of n_2 values of y drawn from $\frac{1}{\sigma} e^{-(y-B_2)/\sigma}\, dy$, the hypothesis we wish to test is that $B_2 = B_1$. Let x_1 be the smallest of the n_1 values of x and y_1 be the smallest of the n_2 values of y, let L be the smallest of the $n_1 + n_2 = N$ values of both x and y. Then the likelihood ratio for this hypothesis is

$$\lambda_3 = \left[\frac{\sum_{i=1}^{n} (x_i - x_1) + \sum_{i=1}^{n_2} (y_i - y_1)}{\sum_{i=1}^{n_1} (x_i - L) + \sum_{i=1}^{n_2} (y_i - L)}\right]^N = \left[\frac{1}{1 + \frac{z}{u}}\right]^N,$$

where

$$\begin{aligned} z &= n_2(y_1 - x_1), \quad \text{if } y_1 > x_1 \\ &= n_1(x_1 - y_1), \quad \text{if } x_1 > y_1, \end{aligned}$$

and

$$u = \sum_{i=1}^{n_1} (x_i - x_1) + \sum_{i=1}^{n_2} (y_i - y_1).$$

The region of acceptance, $\lambda_{3\epsilon} \leq \lambda_3 \leq 1$, is equivalent to the region $0 \leq Z \leq K_3 u$, where K_3 is again a function of α, the level of significance, the exact relation being

$$\int_0^{k_3} \frac{(N-2)\, dt}{(1+t)^{N-1}} = 1 - \alpha, \qquad \text{so} \qquad \frac{1}{(1+k_3)^{N-2}} = \alpha.$$

It is known [3] that u is independent of Z, and that its distribution is

$$\phi_5(u)\, du = \frac{u^{N-3} e^{-u/\sigma}\, du}{\sigma^{N-2}(N-3)!}.$$

The distribution of z is somewhat complicated; but it can be derived by observing that the probability that z lies in any infinitesimal interval $z_1 \pm \frac{1}{2}\, dz_1$ is the sum of the probabilities that $n_2(y_1 - x_1)$ and $n_1(x_1 - y_1)$ lie in that interval and by then using standard methods for finding the distribution of the difference of two variates. For the case $G = B_2 - B_1 \geq 0$, the distribution $f(z)$ of z is

$$(3) \qquad \begin{aligned} f_1(z)\, dz &= \frac{e^{-n_1 G/\sigma}}{(n_1+n_2)\sigma} [n_1 e^{n_1 z/n_2\sigma} + n_2 e^{-z/\sigma}]\, dz, \qquad 0 \leq z \leq n_2 G, \\ f_2(z)\, dz &= \frac{[n_1 e^{n_2 G/\sigma} + n_2 e^{-n_1 G/\sigma}] e^{-z/\sigma}\, dz}{(n_1+n_2)\sigma}, \qquad n_2 G \leq z \leq \infty. \end{aligned}$$

For the case $G \leq 0$, the distribution of z can be derived from (3) by interchanging n_1 and n_2 and putting $-G$ in place of G.

The power function of this test can now be derived. For the case $G \geq 0$, the power function $P(G)$ is

$$P(G) = 1 - \left\{ \int_0^\infty du \int_0^{n_2 G} f_1(z)\phi_5(u)\, dz - \int_0^{n_2 G/k_3} du \int_{k_3 u}^{n_2 G} f_1(z)\phi_5(u)\, dz + \int_{n_2 G/k_3}^\infty du \int_{n_2 G}^{k_3 u} f_2(z)\phi_5(u)\, dz \right\}. \tag{4}$$

Upon integrating out and simplifying, the power function becomes

$$P(G) = \alpha\left(\frac{n_2 e^{-n_1 G/\sigma}}{n_1 + n_2}\right) + I\left[N - 2; \frac{n_2 G}{k_3 \sigma}\right] + \alpha\left(\frac{n_1 e^{n_2 G/\sigma}}{n_1 + n_2}\right)\left\{1 - I\left[N - 2; \frac{n_2 G(1 + k_3)}{k_3 \sigma}\right]\right\} - \frac{n_2}{n_1 + n_2} e^{-n_1 G/\sigma}\left(\frac{n_2}{n_2 - n_1 k_3}\right)^{N-2} I\left[N - 2; \frac{G(n_2 - n_1 k_3)}{k_3 \sigma}\right].$$

The power function when $G \leq 0$ is easily derived from that for $G \geq 0$ by everywhere interchanging n_1 and n_2 and substituting $-G$ for G.

To show that $P(G) > \alpha$ when $G \neq 0$, it is only necessary to show that the derivative $P'(G)$ of the power function is always positive when $G > 0$, and always negative when $G < 0$. It is again considerably simpler to use the expression for $P(G)$ as a double integral. For the case $G > 0$, integrating with respect to z in (4),

$$P(G) = 1 - \frac{n_2}{n_1 + n_2}\left[1 - e^{-G(n_1 + n_2)/\sigma}\right] + \int_0^{n_2 G/k_3} \frac{n_2 e^{-n_1 G/\sigma}}{n_1 + n_2}\left[e^{n_1 z/n_2\sigma} - e^{-z/\sigma}\right]_{k_3 u}^{n_2 G} \phi_5(u)\, du - \int_{n_2 G/k_3}^\infty \frac{(n_1 e^{n_2 G/\sigma} + n_2 e^{-n_1 G/\sigma})}{n_1 + n_2} \cdot \left[-e^{-z/\sigma}\right]_{n_2 G}^{k_3 u} \phi_5(u)\, du,$$

where $[f(x)]_a^b = f(b) - f(a)$. Upon differentiating and simplifying,

$$P'(G) = \frac{n_1 n_2}{(n_1 + n_2)\sigma} \int_0^{n_2 G/k_3} e^{-n_1 G/\sigma}\left[e^{n_1 k_3 u/n_2\sigma} - e^{-k_3 u/\sigma}\right]\phi_5(u)\, du + \frac{n_1 n_2}{(n_1 + n_2)\sigma} \int_{n_2 G/k_3}^\infty e^{-k_3 u/\sigma}\left[e^{n_2 G/\sigma} - e^{-n_1 G/\sigma}\right]\phi_5(u)\, du.$$

Both integrals are easily seen to always be positive, so $P'(G)$ is positive when $G > 0$. In the same manner it can be shown that $P'(G)$ is negative when $G < 0$. Therefore this test is also completely unbiased.

The question of investigating the bias of the likelihood-ratio tests for (a) testing the hypothesis that $\sigma = \sigma_0$ when B is known and (b) testing the hypothesis that $\sigma = \sigma_0$, nothing being known about the value of B, are practically identical with the analogous problems for a normal distribution. The results are also the same, for the λ test for (a) is completely unbiased, while that for (b) is biased.

REFERENCES

[1] J. NEYMAN and E. S. PEARSON, "On the use and interpretation of certain test criteria for purposes of statistical inference," *Biometrika*, Vol. 20a (1928), pp. 221–230.

[2] P. V. SUKHATME, "On the analysis of k samples from exponential populations with special reference to the problem of random intervals," *Stat. Res. Memoirs*, Vol. 1, pp. 94–112.

[3] P. V. SUKHATME, "Tests of significance for samples of the χ^2 population with 2 degrees of freedom," *Annals of Eugenics*, Vol. 8 (1937–38), pp. 54–55.

[4] JOSEPH F. DALY, "On the unbiased character of likelihood-ratio tests for independence in normal systems," *Annals of Math. Stat.*, Vol. 11 (1940), p. 2.

[5] K. PEARSON (Editor), *Tables of the Incomplete Gamma Function*, Biometric Laboratory, London, 1922.

ON THE MATHEMATICALLY SIGNIFICANT FIGURES IN THE SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

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1. **Introduction.** The number of mathematically significant figures in the solution of simultaneous linear equations has received attention from a number of writers [1–6]. It is an important subject, not only in least squares and correlations, but in many other problems of science where simultaneous equations arise: it may not be amiss, therefore, to examine it from a fresh start, particularly since (as will be shown) some of the rules that have been published on it fail in certain frequently occurring circumstances.

2. **Definitions.** Before proceeding into the subject it will be necessary to distinguish between the computer's terms "significant figures" and "determinate significant figures." The former are the figures that compose a number, without the consecutive ciphers that precede or follow them, merely to locate the decimal point. "Determinate significant figures," on the other hand, are figures that are justifiable on computational grounds. From the computer's point of view, the number of significant figures remains independent of what is statistically significant. To avoid confusion in what follows, the term "significant figures" will be used in the computer's sense, and the adjective "determinate" will be supplied where mathematical determinacy is implied.

To avoid prolixity the term "observational error" will include any uncertainty arising either from errors in the observations or from the statistical nature of the problem (e.g. sampling errors, grouping errors, etc.). *The observational error of the result is independent of the particular sequence of computation followed and the accuracy with which it is carried out.*

The term "computational error" will include all the additional uncertainties arising from the approximations occurring in the particular sequence of computation used, including the "rounding off" of the final result. *The computational errors, unlike the observational errors, depend in general upon the sequence of the intermediate steps used in the computation as well as on the number of significant figures to which they are carried.*

3. **Criterion of an adequate computation.** If the number written down at the end of a computation is to serve its purpose the maximum possible computational error must be suitably limited.

A decimal representation of a number containing f significant figures is subject

to an uncertainty (upper limit of absolute error) of 5 in the $(f+1)$th place. It has, therefore, a possible relative (not absolute) error of representation somewhere between $5 \times 10^{-(f+1)}$ and 5×10^{-f}, in magnitude. This relative computational error sets the limit to any valid final rounding off. Regardless of the accuracy to which the intermediate steps of the computation have been carried, this relative computational error introduced by the final rounding off alone must be suitably limited.

In case all of the accuracy obtainable from the data is not needed in the result, the sum of the maximum possible computational error (including the error of the final rounding off) and the maximum possible observational error must be kept below the error which can be tolerated in the result.

In case all of the accuracy obtainable from the data is needed in the result, the maximum possible computational error in the result (including the error of the final rounding off) must be negligible in comparison with the uncertainty (observational error) in the result arising from uncertainty in the data. *Just how small a fraction of the observational error is "negligible" is necessarily a matter of judgment, and will depend upon the nature of the problem.* A computational error that would be wholly negligible in some ordinary computations might be intolerably large in the adjustment of an accurate geodetic survey. In any case the only basis for a valid judgment of the adequacy of the computation lies in a comparison of (i) the maximum possible computational error that can arise in the sequence of computations including the final "rounding off," with (ii) the observational error of the result arising from the observational errors inherent in the data.

4. **Propagation of error in a system of linear equations.** Assume that

$$\sum_t a_{st} x_t = b_s, \qquad s = 1, 2, \cdots, n, \tag{1}$$

is a set of simultaneous linear equations derived in some way from observations and in which the coefficients a_{st} and the absolute terms b_s may all be subject to observational error. If the relative (not absolute) observational error of a quantity q be represented by δ_q it may readily be seen that

$$\begin{cases} \delta x_j = -\sum_h \sum_k (x_k/x_j) A_{hj} a_{hk} \delta a_{hk} + \sum_s (b_s/x_j) A_{sj} \delta b_s \\ \delta \Delta = \sum_h \sum_k A_{hk} a_{hk} \delta a_{hk} \end{cases} \tag{2}$$

where Δ is the determinant of the coefficients a_{hk}, and A_{hk} is the term corresponding to a_{hk} in the reciprocal (not the adjoint) determinant.

5. **Upper limits to observational errors.** The sign and magnitude of the relative errors δa_{hk} and δb_s are unknown, but we shall assume that it is possible

in any problem to assign to them *upper limits*

$$|\delta a_{hk}| \quad \text{and} \quad |\delta b_s|$$

which in magnitude they cannot exceed. If the problem is such that the values of each of the δa_{hk} and the δb_s are wholly independent of each other, it is then possible that their magnitudes may all reach their upper limits $|\delta a_{hk}|$ and $|\delta b_s|$ simultaneously, in which case *upper bounds* of δx_j and $\delta\Delta$ may be placed at

$$(3) \qquad \begin{cases} |\delta x_j| = \sum_h \sum_k |(x_k/x_j)A_{hj}a_{hk}|\,|\delta a_{hk}| + \sum_s |(b_s/x_j)A_{sj}|\,|\delta b_s| \\ |\delta\Delta| = \sum_h \sum_k |A_{hk}a_{hk}|\,|\delta a_{hk}| \end{cases}$$

6. **Indefiniteness of the problem in the general case.** The values of the δa_{hk} and δb_s may not be independent of each other, in which circumstance knowledge of the law of their dependence would make it possible to assign upper limits to the magnitudes of δx_j and $\delta\Delta$. These upper limits can not be larger than the upper bounds shown in equation (3), and in special cases they will be much smaller. Since the dependence of δa_{hk} and δb_s may in general have any form whatever, cases can and will occur in which the upper limits of the relative errors of δx_j and $\delta\Delta$ may have any ratio whatever.

7. **Case of independent errors.** Any general discussion of the errors that can occur in x_j and Δ must be based either on some special assumption or on the limiting assumption that the errors are independent. It is this latter assumption that underlies the usual discussion, and will be the basis of what follows. Equation (3) gives the upper limit to the δx_j and $\delta\Delta$ under these assumptions.

8. **The ratios of $|\delta x_j|$ and $|\delta\Delta|$ are still indefinite in spite of the assumption of independent errors in the coefficients.** However, equation (3) does not determine any definite ratio or inequality between the upper bounds $|\delta x_j|$ and $|\delta\Delta|$. The nature of the observations may be such that some of the errors in the a_{hk} and b_s are very small and some relatively large. Not infrequently it is safe to assume that some of them are free from appreciable error and to ascribe all the error of the x_j to the error in one or two of the a_{hk} or b_s. If any statement of a definite relationship, either as an equality or an inequality between $|\delta\Delta|$ and the $|\delta x_j|$ is valid for all possible sets of linear equations, it must at least hold in the special case in which the errors of all the b_s and the errors of all except one of the a_{hk} are negligible.

If such a statement of a definite general relationship between these upper limits of errors can be made, it must be possible to write down an equation or an inequality between any one of the expressions $|A_{hk}|$ and some or all of the corresponding expressions $|(x_k/x_j)A_{hj}|$, $j = 1, 2, \cdots, n$, that will remain true no matter what be the values of the a_{hk} and the b_s in the original set of simultaneous equations. It is obvious that the ratio of $|A_{hk}|$ and $|(x_k/x_j)A_{hj}|$, $(j \neq k)$, depends upon the values of the a_{hk}, and sets of equations can be found

to give any assigned value to that ratio. It is therefore impossible to state any rule that will restrict the ratio of the relative error of Δ and the relative error of any one of the x_j, valid for all possible sets of linear equations.

9. **Definite statement about the sum of the relative errors in the unknowns.** However, in the summation $\sum_j |\delta x_j|$ there occurs the term corresponding to $j = k$, for which $|(x_k/x_j)A_{kj}| = |A_{kk}|$, so that under the assumption that the a_{hk} and b_h are independent sources of error, we may write the inequality

$$\sum_j |\delta x_j| \not< |\delta \Delta| \tag{4}$$

which states that the sum of the upper bounds to the relative errors of all the x_j cannot be less than the upper bound to the relative error of the determinant Δ. A corresponding statement can easily be proved for the standard deviations.

A limiting case can be constructed in which the inequality (4) reduces to

$$\sum_j |\delta x_j| = |\delta \Delta| \tag{5}$$

and in which all of the $|\delta x_j|$ are equal. For this case,

$$|\delta \Delta| = n\,|\delta x_j| \text{ for all values of } j. \tag{6}$$

If $n \not< 10$ it is obvious that there will be at least one more determinate significant figure in each of the x_j than in the determinant Δ of the coefficients.

It is frequently assumed that the number of determinate significant figures in the solution for any unknown cannot exceed the number of determinate significant figures in the determinant Δ of the coefficients. We see now that this statement can not be generally valid, even under the assumption that the a_{hk} and b_h are independent sources of error. As a matter of fact, it is necessary in some cases to compute some or even all of the unknowns to more significant figures than are determinate in the determinant Δ of the coefficients, if one would retain in the result all the accuracy that is obtainable from the data.

Cases in which the relative observational error of every one of the unknowns is less than the relative error of the determinant Δ probably occur rarely in practice; in fact the only ones that I have seen are those that I constructed purposely to show that such a thing is possible. However, cases in which the relative errors of one or several but not all of the unknowns are much smaller than the relative error of the determinant Δ, occur fairly frequently.

10. **Remarks on the case of "near indeterminacy."** The major interest in curve fitting centers around the condition of "near indeterminacy," i.e., of a small or near vanishing determinant Δ. Even in the circumstance where the relative error of the determinant is much greater than the relative error of some or all of the coefficients and absolute terms, the relative error of one or more of the unknowns may be much smaller than the relative error of the determinant, as may be seen from what follows.

In accurate experimentation the endeavor is, wherever possible, to arrange the experiment so that the quantity sought comes directly from the measurement as represented by an equation such as

$$x = p. \tag{7}$$

However, so ideal an experimental arrangement is rarely if ever possible, and it is a common experience to find that the measurements are represented by an equation such as

$$x + qy + rz + su + \cdots = p, \tag{8}$$

where qy, rz, su, etc., are small corrections that must somehow be evaluated. For simplicity, the discussion will be confined to the almost trivial case

$$x + qy = p. \tag{9}$$

Not infrequently the only way the correction can be evaluated is to rearrange the conditions of the experiment so that another equation is obtained in the form

$$x + q'y = p'. \tag{10}$$

Sometimes the nature of the experiment is such that it is not possible to change the coefficient of y by more than a small amount, under which conditions

$$q' = q(1 + \beta), \tag{11}$$

and

$$p' = p(1 + \alpha), \tag{12}$$

where β and α are small in comparison with 1. The solution of equations (9) and (10) now gives

$$x = \frac{\begin{vmatrix} p & q \\ p' & q' \end{vmatrix}}{\begin{vmatrix} 1 & q \\ 1 & q' \end{vmatrix}} = \frac{pq' - p'q}{q' - q} = p(1 - \alpha/\beta). \tag{13}$$

The quantity $q' - q$ seen in the denominator of this equation is the determinant Δ of the coefficients, and by equation (11) its value is βq. Since βq is assumed to be small here, the solution for x encounters a near vanishing denominator. It would, however, be wrong to assume that the number of determinate significant figures in x that can be obtained by solving the equations is necessarily limited to the number of determinate significant figures in the denominator Δ.

If the experimenter has been fortunate in finding suitable experimental conditions, the denominator $\Delta = \beta q$, although small in comparison with either q' or q, will still not cause difficulty. It will be observed that the coefficients of q' and q in the denominator are equal (both being unity). Now if the coefficients p and p' in the numerator are nearly enough equal, so that q' and q occur in both

numerator and denominator so nearly proportionally that the uncertainties in q and q' produce nearly compensating errors in both numerator and denominator, then x will be given to more determinate significant figures than are found in the denominator Δ. It can then be said that the experiment is successful in evaluating the correction term qy in equation (9).

On the other hand, in less fortunate circumstances, to the exasperation of the experimenter, the denominator $\Delta = q' - q = \beta q$ is not only small, but p' and p, although still nearly equal, differ enough so that the errors in q' and q are not compensated by the nearly equal coefficients in the numerator. The experiment will then fail to improve the approximation p for x by failing to evaluate the small correction qy in equation (9). This would be an inherent defect in the experiment and could not be removed by any manner of computation.

The same conclusion would of course be drawn from the coefficient of p (viz., $1 - \alpha/\beta$) at the extreme right of equation (13). It is not the size of β that alone determines the number of determinate significant figures in x, it is rather the ratio between α and β. In the fortunate experimental circumstances described above, the near equality of p' and p offsets the near equality of q' and q by reducing the term α/β to a value small compared with unity; the term α/β, being small, acts to reduce the effect of the uncertainties in q and q' (i.e., in q and β) in the evaluation of x. On the other hand, in less fortunate circumstances, the correction term α/β can not now shield x from the uncertainties in q and q' since the relative difference α between p and p' is not small enough to reduce α/β to innocuity.

11. **Numerical illustration of compensating errors.** As a "horrible example" especially constructed to emphasize the theoretical possibilities, take the following special case—

$$\text{(14)} \qquad \begin{cases} 1000.10000x + 10.00000y = 1010.10000 \\ 1000.00000x + 10.00000y = 1010.00000 \end{cases}$$

wherein it is assumed that the coefficients and the absolute terms (assumed to be derived from the observational data) are all correct to the fifth decimal place as given, and no closer estimate of their errors is possible. So far as known, the upper limit to the absolute observational error of each is then the same, i.e. 5×10^{-6}, but the coefficients of x (a_{11} and a_{21}), and the absolute terms (b_1 and b_2), all have nine determinate significant figures, while the coefficients of y (a_{12} and a_{22}), have only seven. Thus,

$$|\delta a_{11}| \not> 5 \times 10^{-9}, \qquad |\delta a_{21}| \not> 5 \times 10^{-9}, \qquad |\delta b_1| \not> 5 \times 10^{-9},$$
$$|\delta b_2| \not> 5 \times 10^{-9},$$

but

$$\text{(15)} \qquad |\delta a_{12}| \not> 5 \times 10^{-7}, \qquad |\delta a_{22}| \not> 5 \times 10^{-7},$$

and $x = 1$, $y = 1$, $\Delta = 1$, whereupon a substitution of values from (15) into (3) gives the inequalities

$$(16) \quad |\delta x| \gg 3 \times 10^{-4}, \qquad |\delta y| \gg 3 \times 10^{-2}, \qquad |\delta \Delta| \gg 1.01 \times 10^{-2}.$$

So far as known, the determinant Δ may thus be in error by as much as 1 per cent, and y by as much as 3 per cent, yet x is known closer than 1/30th per cent. Here the value of the unknown x cannot be adequately represented by less than four significant figures, and might even require five, in spite of the fact that neither Δ nor y requires more than three significant figures to represent all that is certainly known about them.

The reason for this disparity in relative errors can be more easily seen by substituting numerical values for all the coefficients in the expression for x except a_{12} and a_{22}. The possible relative errors of a_{12} and a_{22} are, as noted above, about 100 times as great as the possible relative errors of a_{11}, a_{21}, b_1, and b_2, and are the controlling errors in Δ. In the solution

$$(17) \qquad x = \frac{1010.10000a_{22} - 1010.00000a_{12}}{1000.10000a_{22} - 1000.00000a_{12}},$$

however, both a_{12} and a_{22} occur in both numerator and denominator, and moreover the coefficient of each in the numerator is nearly equal to its coefficient in the denominator, so that a change in either a_{12} or a_{22} changes both numerator and denominator nearly proportionally, with the result that their ratio x is known much more accurately than either the numerator or the denominator Δ.

This kind of compensation of errors in a computation is not confined to the solution of simultaneous equations (and it is not an infrequent occurrence in other computations). This is one of the many reasons why it is impossible to give general rules for the retention of significant figures that will be valid for all types of computations.

12. **Geometrical analogy.** Moulton [4] illustrated his reasoning by the following geometrical analogy. The solution of three linear equations is equivalent to finding the point of intersection of three planes. When the determinant of the coefficients is small in comparison with the coefficients themselves, these planes are either nearly parallel, or the line of intersection of any two of them is nearly parallel to the third. In these cases small uncertainties in the location of any one of the planes correspond to large uncertainties in the position of their point of intersection.

In the first circumstance the planes might all be nearly parallel to one of the three coordinate planes, with the result that large uncertainty would afflict the value of the determinant and two of the unknowns, the third being much more accurately determined.

In the second circumstance, the line of intersection of two of the planes might be nearly parallel to one of the coordinate axes. When that happens, large un-

certainty will afflict the value of the determinant, but only one of the unknowns, the other two being much more accurately determined.

This geometrical analogy can be extended to cover simultaneous equations with any number of unknowns. Near-vanishing of the determinant Δ of the coefficients necessarily implies relatively large uncertainties in the determinant and also in at least one of the unknowns, but not necessarily in all of them. These are, of course, very special cases, but, as noted above, they are of frequent occurrence in actual problems.

13. **Evaluation of computational error.** The relative computational error in x_j must be kept within certain definite limits which depend upon the particular problem to be solved (section 3). To do this it is necessary to be able to calculate an upper bound to the relative computational error inherent in any particular sequence of computations.

In many computations it is easy to write down a simple formula that will set an upper bound to the relative computational error involved in that particular sequence. This formula contains numbers f_1, f_2, f_3, etc., each representing the number of significant figures accurately computed at some particular step. Once a simple formula for relative computational error is written down, it is easy to choose values of f_1, f_2, f_3, etc. that will give an upper bound to the relative computational error not larger than the permissible limit of maximum possible computational error outlined in section 3. This method of determining an upper bound of the relative computational error should be used whenever such a simple formula can be found. For example, to compute x from equation (13) we may use the following sequence: $r_1 = q' - q$, $r_2 = r_1/q = \beta$, $r_3 = p' - p$, $r_4 = r_3/p = \alpha$, $r_5 = r_4/r_2 = \alpha/\beta$, $r_6 = 1 - r_5 = 1 - \alpha/\beta$, $r_7 = pr_6 = p(1 - \alpha/\beta) = x$. x may then be written as a function of these partial results, viz.:

$$x = r_7 = pr_6 = p(1 - r_5) = p(1 - r_4/r_2) = p(1 - r_3/pr_2) = p(1 - r_4q/r_1). \tag{18}$$

Applying first order error theory we find

$$|\epsilon(x)| \leqq \left|\frac{\alpha/\beta}{1 - \alpha/\beta}\right| \{|\epsilon(r_1)| + |\epsilon(r_2)| + |\epsilon(r_3)| + |\epsilon(r_4)| + |\epsilon(r_5)| + |\epsilon(r_6)| + |\epsilon(r_7)|\} \tag{19}$$

where $\epsilon(r_i)$ represents the relative error in r_i arising from the computation by which r_i was determined from the preceding partial results, $r_1, r_2, \cdots, r_{i-1}$, and $\epsilon(x)$ is the total relative computational error in x when so computed. It is easy to keep $\epsilon(x)$ within any desired limits by suitably limiting each error term of (19). Since a computation accurate to f significant figures involves a relative computational error not greater than 5×10^{-f}, any desired limits can then be set to each error term of (19) by a proper choice of the number of significant figures that should be carried in that step.

Unfortunately there seem to be no reasonably simple formulae for determining upper bounds of the relative computational errors that arise in the solution of simultaneous linear equations in more than two variables. This does not absolve the computer from the necessity of ensuring that his computational errors are suitably limited.

The method I have found most economical is to carry the solution of simultaneous linear equations to the capacity of the machine, and as each partial result r_i is obtained, write it as

$$r_i(1 \pm \epsilon_i),$$

where r_i is the value actually found and ϵ_i is a positive number representing the accumulation of uncertainty introduced by all preceding steps in the computation. At the end of the computation each of the unknowns is found in the form

$$x_j(1 \pm \epsilon_j), \tag{20}$$

where x_j represents the value found and ϵ_j is the upper bound of the relative computational error in x_j.

A comparison of ϵ_j with the upper bound of the observational error $|\delta x_j|$ of equation (3) will then indicate whether the computation is adequate. If the comparison shows that the computation was inadequate, it will show in which steps the number of significant figures f_i was too small, and by how much. The computer can recompute, carrying these steps to the requisite number of figures with the assurance that his recomputation will then be adequate. The comparison will further indicate in which steps if any the number of significant figures f_i was larger than necessary.

When a computer has thus set suitable upper bounds to the relative computational error in the solution of a set of linear equations, he is in a position to plan solutions of future similar sets so as to perform his computations more economically and yet safely. This is especially true when the solution of simultaneous linear equations arises week after week in routine testing.

14. **Conclusions.** Summary rules have been published, purporting to be safe guides to computers in avoiding needless work, and ensuring that the computations are carried to a sufficient degree of accuracy. Many of them are useful guides for certain types of computation and for limited ranges of the numerical values entering into the computation, but none of those that I have seen can be used generally. The only safe rule, where the matter is of importance, is to calculate the maximum possible computational error that can enter in the particular sequence of computation followed, and make sure that it is kept within the necessary limits.

It is sometimes necessary to carry the intermediate steps of a computation to many significant figures beyond the significant figures given in the data, or kept in the result. The relative error of one of the unknowns may be very much smaller than the relative errors of the data from which it is computed, while the

relative error of another of the unknowns may be larger. The methods of ensuring that the computations are adequate are outlined in section 13.

For the best sequence to follow in the elimination of the unknowns, I shall pass along a suggestion of Dr. W. Edwards Deming which he gave in one of our discussions of this subject. I venture to pass it along, because it has worked in every special case that I have constructed in an attempt to prove that it does not hold generally. If ever the suggestion fails, the computer may change the sequence; but in any case he is obliged, as stated above, to calculate the maximum possible computational error that can enter into his calculations. Dr. Deming's suggestion is this: "To evaluate some but not all of the unknowns to the highest possible computational accuracy, retaining as few significant figures as possible in the intermediate steps, solve the equations by successive elimination, eliminating first and evaluating last the unknowns of greatest inherent relative accuracy."

15. **Summary.** Expressions are given for the maximum observational error in the unknowns of a system of simultaneous linear equations, in terms of the relative errors of the coefficients and absolute terms therein. In order to extract all the information possible from a system of linear equations representing observational results, it is not sufficient in general to assume that the relative errors in the unknowns are as large as the relative error in the determinant of the system. In many problems the computation of some of the unknowns must therefore be carried to more significant figures than are determinate in the determinant of the system. Methods are outlined for evaluating computational error in the solution of linear equations to ensure that the computations are adequate.

In conclusion I wish to express my thanks to Dr. W. Edwards Deming who has given much of his time to assist me in the preparation of this paper. He has made valuable suggestions on the material to be included and the general manner of presentation. In addition he has criticized the manuscript in detail and assisted in the final revision.

REFERENCES

[1] Edward B. Roessler, *Science*, Vol. 84, (1936), pp. 289–290.
[2] Joseph Berkson, ibid., p. 437.
[3] P. J. Rulon, ibid., pp. 483–484.
[4] F. R. Moulton, ibid., pp. 574–575.
[5] W. Edwards Deming, *Science*, Vol. 85 (1937), pp. 451–454. *Least Squares* (Department of Agriculture Graduate School, 1938), pp. 105, 111, 121, 135.
[6] I. M. H. Etherington, *Edinb. Math. Soc. Proc.*, Vol. 3 (1932), Part 2, pp. 107–117.

ON MECHANICAL TABULATION OF POLYNOMIALS

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1. Introduction. The purpose of this paper is to show how automatic accounting machines, which have been used previously in evaluating such quantities as Σx^n and $\Sigma x^{n-1}y$, may be used in the preparation of mathematical tables of integral powers, of polynomials, and of functions which can be approximated by polynomials. These tables may be prepared for any desired intervals of the argument such as 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{2}$, $\frac{1}{3}$, etc.

The method is an adaptation of the general theory of "cumulative" or "progressive" totals which has proved useful in computing moments and product moments both with and without accounting machines. The reader unfamiliar with the mathematical method and its machine applications might refer to such presentations as those of Hardy [1], Mendenhall and Warren [2, 3], Razram and Wagner [4], Brandt [5], and Dwyer [6, 7]. The main feature of the method is the computation of summed products or of summed powers by means of successive cumulated additions. It is shown in this paper how it is possible to use this same process in constructing tables of powers and tables of polynomials.

2. The Cumulative Formulas. If the numbers F_x are defined and finite for $x = 1, 2, 3, \cdots, (a-1), a$, and if these values of F_x are cumulated for $x = a$, $x = a - 1$, etc., then the value in the row headed by $x = 1$ can be written as ${}^1T_1$. If these cumulations are cumulated successively with the superscript indicating the order of the cumulation and the subscript indicating the value of x which heads the row, then

$$ {}^2T_1 = \Sigma x F_x, \qquad {}^3T_1 = \Sigma \frac{(x+1)x}{2!} F_x, \qquad {}^3T_2 = \Sigma \frac{x(x-1)}{2!} F_x, $$

$$ {}^4T_1 = \Sigma \frac{(x+2)(x+1)x}{3!} F_x $$

and in general for $i < j$,

$$ {}^jT_i = \Sigma \frac{[x+j-(i+1)]^{(j-1)}}{(j-1)!} F_x. \tag{1} $$

Formula (1) is basic to much of the previous work involving cumulative totals. Various authors have studied such important special cases as (A) where F_x equals the frequency function f_x, (B) where $F_x = xf_x$, and (C) where F_x equals the sum of all the values of y having the same x value. These special cases have been found very useful in computing moments and product moments.

The moments may be expressed in terms of the cumulations in a variety of ways. The diagonal formulas have the differences of zero as coefficients and are expressed in terms of ${}^1T_1$, ${}^2T_1$, ${}^3T_2$, ${}^4T_2$, ${}^5T_4$, etc. The columnar formulas, whose coefficients have been recently studied [6, 7], are expressed in terms of cumulations of the same order, ${}^jT_x$, with j fixed. Razram and Wagner [4] have given formulas which utilize the entries of different rows and different columns but which demand fewer entries for the formulas. Razram and Wagner worked out the formulas through $\Sigma x^4 f_x$ but the argument holds for $\Sigma x^4 F_x$. For purposes of comparison the values of $\Sigma x^i F_x$, $i = 0, 1, 2, 3, 4$, as they appear in the diagonal, columnar, and Razram-Wagner systems are presented in Table I.

TABLE I

Values of $\Sigma x^i F_x$ for $i = 0, 1, 2, 3, 4$.

F_x	Diagonal	Columnar	Razram-Wagner
ΣF_x	${}^1T_1$	${}^1T_1$	${}^1T_1$
$\Sigma x F_x$	${}^2T_1$	${}^2T_1$	${}^2T_1$
$\Sigma x^2 F_x$	${}^2T_1 + 2{}^3T_2$	${}^3T_1 + {}^3T_2$	${}^3T_1 + {}^3T_2 = {}^3T_{1+2}$
$\Sigma x^3 F_x$	${}^2T_1 + 6{}^3T_2 + 6{}^4T_3$	${}^4T_1 + 4{}^4T_2 + {}^4T_3$	${}^2T_1 + 6{}^4T_2$
$\Sigma x^4 F_x$	${}^2T_1 + 14{}^3T_2 + 36{}^4T_3 + 24{}^5T_4$	${}^5T_1 + 11{}^5T_2 + 11{}^5T_3 + {}^5T_4$	${}^3T_{1+2} + 12{}^5T_{2+3}$

In developing the theory of the later sections of this paper I have developed further formulas of the type shown by Razram and Wagner since these formulas have fewer terms than do those of the other systems and the coefficients are factorable by $(j-1)!/2$. These formulas for $\Sigma x^s F_x$, with s even, feature such terms as ${}^3T_1 + {}^3T_2 = {}^3T_{1+2}$, ${}^5T_{2+3}$, etc., so that there are two entries from the same column. For the purposes of this paper it is preferable to have a single entry from each column and this situation results from continued application of the formula

$$(2) \qquad {}^jT_{i+(i+1)} = {}^jT_i + {}^jT_{i+1} = {}^{j-1}T_i + 2\,{}^jT_{i+1}.$$

The formulas for $\Sigma x^s F_x$ with $s \leqq 12$ are given. The alternative forms are given for the formulas involving even values of s.

$$\Sigma F_x = {}^1T_1, \qquad \Sigma x F_x = {}^2T_1, \qquad \Sigma x\,{}^2F_x = {}^3T_1 + {}^3T_2 = {}^3T_{1+2} = {}^2T_1 + 2\,{}^3T_2,$$

$$\Sigma x\,{}^3F_x = {}^2T_1 + 6\,{}^4T_2, \qquad \Sigma x\,{}^4F_x = {}^3T_{1+2} + 12\,{}^5T_{2+3}$$

$$= {}^2T_1 + 2\,{}^3T_2 + 12\,{}^4T_2 + 24\,{}^5T_3,$$

$$\Sigma x\,{}^5F_x = {}^2T_1 + 30\,{}^4T_2 + 120\,{}^6T_3,$$

$$\Sigma x\,{}^6F_x = {}^3T_{1+2} + 60\,{}^5T_{2+3} + 360\,{}^7T_{3+4}$$

$$= {}^2T_1 + 2\,{}^3T_2 + 60\,{}^4T_2 + 120\,{}^5T_3 + 360\,{}^6T_3 + 720\,{}^7T_4,$$

$$\Sigma x\,{}^7F_x = {}^2T_1 + 126\,{}^4T_2 + 1680\,{}^6T_3 + 5040\,{}^8T_4,$$

$$\Sigma x\,{}^8F_x = {}^3T_{1+2} + 252\,{}^5T_{2+3} + 5040\,{}^7T_{3+4} + 20160\,{}^9T_{4+5}$$

$$= {}^2T_1 + 2\,{}^3T_2 + 252\,{}^4T_2 + 504\,{}^5T_3 + 5040\,{}^6T_3 + 10080\,{}^7T_4$$

$$+ 20160\,{}^8T_4 + 40320\,{}^9T_5,$$

$$(3)\quad \Sigma x\,{}^{9}F_x = {}^{2}T_1 + 510\,{}^{4}T_2 + 17640\,{}^{6}T_3 + 151200\,{}^{8}T_4 + 362880\,{}^{10}T_5,$$

$$\Sigma x\,{}^{10}F_x = {}^{3}T_{1+2} + 1020\,{}^{5}T_{2+3} + 52920\,{}^{7}T_{3+4} + 604800\,{}^{9}T_{4+5} + 1814400\,{}^{11}T_{5+6}$$

$$= {}^{2}T_1 + 2\,{}^{3}T_2 + 1020\,{}^{4}T_2 + 2040\,{}^{5}T_3 + 52920\,{}^{6}T_3 + 105840\,{}^{7}T_4$$

$$+ 604800\,{}^{8}T_4 + 1209600\,{}^{9}T_5 + 1814400\,{}^{10}T_5 + 3628800\,{}^{11}T_6,$$

$$\Sigma x\,{}^{11}F_x = {}^{2}T_1 + 2046\,{}^{4}T_2 + 168960\,{}^{6}T_3 + 3160080\,{}^{8}T_4 + 19958400\,{}^{10}T_5$$

$$+ 39916800\,{}^{12}T_6,$$

$$\Sigma x\,{}^{12}F_x = {}^{3}T_{1+2} + 4092\,{}^{5}T_{2+3} + 506880\,{}^{7}T_{3+4} + 12640320\,{}^{9}T_{4+5}$$

$$+ 99792000\,{}^{11}T_{5+6} + 239500800\,{}^{13}T_{6+7}$$

$$= {}^{2}T_1 + 2\,{}^{3}T_2 + 4092\,{}^{4}T_2 + 8184\,{}^{5}T_3 + 506880\,{}^{6}T_3 + 1013760\,{}^{7}T_4$$

$$+ 12640320\,{}^{8}T_4 + 25280640\,{}^{9}T_5 + 99792000\,{}^{10}T_5$$

$$+ 199584000\,{}^{11}T_6 + 239500800\,{}^{12}T_6 + 479001600\,{}^{13}T_7.$$

The derivation of these formulas is obtained with the use of (1), with the use of

$$(4)\quad {}^{j}T_i = {}^{j}T_{i+1} + {}^{j-1}T_i,$$

and with the use of formulas of lower order. For example we have from (1)

$$\Sigma \frac{(x+4)(x+3)(x+2)(x+1)x}{120} F_x = {}^{6}T_1$$

so that

$$\Sigma x\,{}^{5}F_x = 120\,{}^{6}T_1 - 10\,\Sigma x\,{}^{4}F_x - 35\,\Sigma x\,{}^{3}F_x - 50\,\Sigma x\,{}^{2}F_x - 24\,\Sigma x F_x$$

which after substitution of $\Sigma x\,{}^{4}F_x$, $\Sigma x\,{}^{3}F_x$, etc. and simplification results in the value ${}^{2}T_1 + 30\,{}^{4}T_2 + 120\,{}^{6}T_3$.

3. Tables of powers. If $F_x = 1$ when $x = a$, but is zero otherwise then $\Sigma x\,{}^{s}F_x$ is equal to a^s. It follows that the value of a^s can be obtained from the successive cumulations of this F_x with the use of (3). For example in Table II

TABLE II

Cumulations of $F_x = 1$, *when* $x = 6$,
0, *when* $x \neq 6$.

a	x	F_x	^1T	^2T	^3T	^4T	^5T
1	6	1	1	1	1	1	1
2	5	0	1	2	3	4	5
3	4	0	1	3	6	10	15
4	3	0	1	4	10	20	35
5	2	0	1	5	15	35	70
6	1	0	1	6	21	56	126
7		0	1	7	28	84	210
8		0	1	8	36	120	330

$$6^2 = {}^2T_1 + 2\,{}^3T_2 = 6 + 2(15) = 36,$$

$$6^3 = {}^2T_1 + 6\,{}^4T_2 = 6 + 6(35) = 216,$$

$$6^4 = {}^2T_1 + 2\,{}^3T_2 + 12\,{}^4T_2 + 24\,{}^5T_3 = 6 + 2(15) + 12(35) + 24(35) = 1296.$$

The values of ${}^2T_1$, ${}^3T_2$, ${}^4T_2$ and ${}^5T_3$ for $a = 6$ are italicized in Table II.

To get the values of 5^2, 5^3, 5^4, etc. it would be necessary to start to cumulate from $x = 5$. Now since the values of ${}^1T_i$ are unity, it follows that the values for $a = 5$ can be found by taking the entries above those for $a = 6$. Thus ${}^2T_1 = 5$, ${}^3T_2 = 10$, ${}^4T_2 = 20$, ${}^5T_3 = 15$ with $5^2 = 5 + 2(10)$, $5^3 = 5 + 6(20)$, $5^4 = 5 + 2(10) + 12(20) + 24(15)$. It is evident in general that the values for any a^2, a^3, a^4 can be obtained by taking the row headed by a as the bottom row. Thus using $a = 8$, we have $8^2 = 8 + 2(28)$, $8^3 = 8 + 6(84)$, etc. It then appears that we may omit the x column of Table II and consider the cumulations to be ascending cumulations for a rather than descending cumulations for x.

A more satisfactory course is to cumulate the coefficients so as to eliminate the multiplications. Thus the value of $6\,{}^jT_i$ could be obtained without multiplication by cumulating 6, 0, 0, 0, 0 $\cdots$ rather than 1, 0, 0, 0, $\cdots$. Several cumulations may be carried on at the same time so that the additions are not necessary and the tabulation results in a table of the desired powers.

In preparation of a power table, the formulas (3) become a series of instructions on the way in which we are to do the cumulating. For instance the formula:

$$x^7 = 5040\,{}^8T_4 + 1680\,{}^6T_3 + 126\,{}^4T_2 + {}^2T_1,$$

tells us that to form a table of the seventh power we must cumulate[1] the coefficient 5040 eight times; add in the coefficient 1680 when there are six operations; the coefficient 126 when there are four; and the coefficient 1 when there are two remaining. A change in subscript tells us that the coefficient when first included forms a separate total ahead of the ones already partly figured. When the subscript does not change, the coefficient is to be included in the first summary card total. The final cumulating operation prints the actual table.

To prepare a power table by machine we secure a set of cards punched all alike with the numbers from 1 to 9 punched diagonally in successive columns across the card. The machine is wired to add the coefficient of the highest term by selecting the proper digits from the diagonals, cumulate after each card and summary punch each total. This way of starting saves one cumulation. The summary cards are cumulated repeatedly in the same manner until the number of operations indicated by the highest term is completed. When the number of operations remaining equals j of another term ${}^jT_i$, a card for the coefficient of that term is included in the tabulation ahead of the summary cards. This automatically adds the new coefficient to each term of the series. When the subscript i in ${}^jT_i$ changes, the new coefficient card must form a separate total;

[1] This operation is generally known as *progressive totalling* in machine operation.

when it does not change, the coefficient card must tabulate in the first summary card total.

To illustrate the tabulation of power tables, the formula for the cube table is— $x^3 = 6\,{}^4T_2 + {}^2T_1$.

The successive operations yield the following table:

TABLE III

x	*Operation number*			
	1	2	3	4:x^3
1	0	0	1	1
2	6	6	7	8
3	6	12	19	27
4	6	18	37	64
5	6	24	61	125
6	6	30	91	216
7	6	36	127	343
8	6	42	169	512
9	6	48	217	729
10	6	54	271	1000

In actual machine work, operation 1 can be omitted and work begun with operation 2. The machine is set to add the coefficient 6 of the highest term from each card and an accumulated total is printed and punched for each card tabulated, giving the results shown under operation 2. An additional card is punched for the coefficient of the second term, 1, and placed ahead of the cards produced in operation 2. The cumulation and punching is repeated, giving the results shown under operation 3. The summary cards from this operation are cumulatively tabulated, giving the results shown under operation 4, which is the table of cubes desired.

Similarly, for a table of the fourth power, the formula $x^4 = 24\,{}^5T_3 + 12\,{}^4T_2 + 2\,{}^3T_2 + {}^2T_1$ indicates the following operations—

TABLE IV

x	*Operation number*				
	1	2	3	4	5:x^4
1	0	0	0	1	1
2	0	12	14	15	16
3	24	36	50	65	81
4	24	60	110	175	256
5	24	84	194	369	625
6	24	108	302	671	1296
7	24	132	434	1105	2401
8	24	156	590	1695	4096
9	24	180	770	2465	6561
10	24	204	974	3439	10000
11	24	228	1202	4641	14641
12	24	252	1454	6095	20736

Note in operation 3 where the subscript does not change, the coefficient 2 is added to the first card punched by the machine, while in operation 4 where it changes, the coefficient 1 appears as a separate total.

4. Tables of polynomials. To tabulate values of $f(x) = a + bx + cx^2 \cdots$ (where $a, b, c, \cdots$, are positive or negative coefficients) the method is similar to that of preparing power tables except that the coefficients to be added are determined by multiplying the coefficients of the formulas for the different powers by the values a, b, c etc., adding the coefficients of like terms in the various formulas, and using these resultant coefficients in place of the simple coefficients used in the power tables. Thus if we wish to tabulate values of $f(x) = 4 + 3x + 2x^2 + x^5$ the coefficients are found as follows:

$$\begin{aligned}
4x^0 &= 4\,{}^1T_0 \\
+\,3x &= +\,3\,{}^2T_1 \\
+\,2x^2 &= +\,2\,{}^2T_1 + 2\cdot 2\,{}^3T_2 \\
+\,x^5 &= +\,{}^2T_1 + 30\,{}^4T_2 + 120\,{}^6T_3 \\
\hline
f(x) &= 4\,{}^1T_0 + 6\,{}^2T_1 + 4\,{}^3T_2 + 30\,{}^4T_2 + 120\,{}^6T_3
\end{aligned}$$

This equation gives instructions to perform six operations with 120 as coefficient; adding the coefficient 30 as a separate total when there are 4 operations remaining; adding 4 to the first summary card total when there are 3 operations; adding 6 as a separate total when there are 2 operations remaining; and adding 4 on the last operation.

The first few totals appear thus—

TABLE V

x	Operation number					
	1	2	3	4	5	6:$f(x)$
0						4
1	0	0	0	0	6	10
2	0	0	30	34	40	50
3	120	120	150	184	224	274
4	120	240	390	574	798	1072
5	120	360	750	1324	2122	3194
6	120	480	1230	2554	4676	7870
7	120	600	1830	4384	9060	*16930*
8	120	720	2550	6934	*15994*	32924
9	120	840	*3390*	*10324*	26318	59242
10	*120*	*960*	4350	14674	40992	100234

It is not necessary to confine these tables to values for whole numbers, as we can tabulate equally well values of $f(x)$ for intervals of x of .1, .01 or .001 or $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc. In this case, before combining formulas for different powers we multi-

ply both sides by the desired interval raised to the power to which x is raised in that particular formula, then add like terms as before.

To tabulate the previous example in $.1x$ intervals we proceed as follows:

$$
\begin{aligned}
4x^0 &= 4.000\ {}^1T_0 \\
3x/10 &= +.3\ {}^2T_1 \\
2(x/10)^2 &= +.02\ {}^2T_1 + .04\ {}^3T_2 \\
(x/10)^5 &= +.00001\ {}^2T_1 + .00030\ {}^4T_2 + .00120\ {}^6T_3 \\
\hline
f(x) &= 4\ {}^1T_0 + .32001\ {}^2T_1 + .04\ {}^3T_2 + .00030\ {}^4T_2 + .00120\ {}^6T_3
\end{aligned}
$$

TABLE VI

x	*Operation number*					
	1	2	3	4	5	6:$f(x)$
1	0	0	0	0	.32001	4.32001
2	0	0	.0003	.0403	.36031	4.68032
3	.0012	.0012	.0015	.0418	.40211	5.08243
4	.0012	.0024	.0039	.0457	.44781	5.53024
5	.0012	0036	.0075	.0532	.50101	6.03125
6	.0012	.0048	.0123	.0655	.56651	6.59776
7	.0012	.0060	.0183	.0738	.64031	7.23807
8	.0012	.0072	.0255	.0993	.73961	8.07768
9	.0012	.0084	.0339	.1332	.87281	8.95049
10	.0012	.0096	.0435	.1767	1.04951	10.00000

Where any coefficients are negative in the equations expressed in ${}^jT_i$ terms, they are simply added in as minus figures.

To round off the preceding function to 3 decimal places, we add 5 to the constant term ${}^1T_0$ in the position to the right of the last decimal retained, i.e. in this case the 4th decimal place. The constant term is then 4.0005.

Exact	*Counter reads*	*Prints*
4.32001	4.32051	4.320
4.68032	4.68082	4.680
5.08243	5.08293	5.082
5.53024	5.53074	5.530
6.03125	6.03175	6.031
6.59776	6.59826	6.598
7.23807	7.23857	7.238
8.07768	8.07818	8.078
8.95049	8.95099	8.950
10.00000	10.00050	10.000

5. Automatic calculation of polynomial coefficients. Frequently when polynomials are being evaluated, the process of forming the coefficients can be

performed automatically from a punched-card table. Such a table consists of a set of cards for each power x^s containing the multiples of all the coefficients of each of the terms ${}^jT_i$ in the formula (3) for that power. These multiples are 1, 2, 3, 4, $\cdots$, 9; 10, 20, 30, 40, $\cdots$, 90; 100, 200, $\cdots$, 900; 1000, 2000 etc., and may be produced automatically by making a linear table of each coefficient in the manner described in this paper. Each card is punched with the information called for by the heading of the following card form:

s	j	i	multiple	coeff. $\times$ multiple
07	06	03	00005	008400

The particular figures indicated are those which would be punched for the term $5(1680){}^6T_3$ in the representation of $5x^7$ according to formula (3).

The table is used by withdrawing the cards for the coefficients a, b, c, d, etc. of the desired polynomial. For instance, if one of the polynomial coefficients is $14485\,x^7$, we select from the x^7 section of the table all cards containing the multiples 10000, 4000, 400, 80, and 5. In the x^7 table there are 4 cards for each multiple, one each for terms ${}^8T_4$, ${}^6T_3$, ${}^4T_2$, and ${}^2T_1$. These cards are combined with the cards selected for the other coefficients of the polynomial and sorted to bring all cards for each ${}^jT_i$ together. The cards for each term ${}^jT_i$ are then automatically added on the electric accounting machine.

6. Subdividing tables. In preparing tables it may be desired to prepare the table in more detail at certain points, giving values of the function at 1/10, 1/20, 1/50, or 1/100, etc., of the interval of the rest of the table. This may readily be done by recalculating the coefficients of the cumulative terms, and using these values in the same manner as the original ones.

There are many formulas for the determination of the subdivided differences given in various texts on interpolation, such as those given by Comrie [8] and Bower [9]. One effective method is to use formulas (3) to calculate the subdivided differences. The values called for in the formula for the highest power are taken from the table of the function at the regular interval, giving effect to the rule involving subscripts. These coefficients are reduced by an amount sufficient to cancel the coefficient of the highest cumulative term, and the coefficients of the remaining cumulative terms are reduced in proportion according to formula (3) for the highest power. Usually the coefficient of the highest term of the formula will divide evenly into the coefficient taken from the table, and the other reductions are calculated by multiplying this result by the other coefficients of the formula. The highest remaining coefficient is then reduced by an amount sufficient to cancel itself, and, by use of the formula (3) for the power whose highest cumulative term matches the highest remaining coefficient, the reduction to the remaining cumulative terms is calculated and subtracted.

The highest remaining coefficient is reduced in a like manner, and this process is continued until all the cumulative coefficients have been analyzed.

The partial cumulative coefficients thus computed are multiplied by the desired subdivision $1/m$ raised to the power of the corresponding formula (3), and recombined to form the new coefficients, as shown in the example below. In taking values from the table, when the subscript does not change, the tabular value must be reduced by the amount of the higher coefficient with the same subscript, to give effect to the rule that the coefficients in such cases are increments (see last example in section 3).

To subdivide the polynomial of section 4 at $x = 7.0$, we take the italicized values from Table V starting at $f(7)$ as ${}^1T_0$, and proceed as follows:

	${}^6T_3$	${}^5T_3$	${}^4T_2$	${}^3T_2$	${}^2T_1$	${}^1T_0$
		960	-	10324		
From Table V	120	−120	3390	−3390	15994	16930
$F(x)$.....	120	840	3390	6934	15994	16930
ax^5	120		30		1	
		840	3360	6934	15993	
bx^4		840	420	70	35	
			2940	6864	15958	
cx^3			2940		490	
				6864	15468	
dx^2				6864	3432	
					12036	
ex.....					12036	
						16930

If the interval is 1/10 we have:

	${}^6T_3$	${}^5T_3$	${}^4T_2$	${}^3T_2$	${}^2T_1$	${}^1T_0$
$x^5/10^5$ =	.00120		.00030		.00001	
$35x^4/10^4$ =		.0840	.04200	.0070	.00350	
$490x^3/10^3$ =			2.94000		.49000	
$3432x^2/10^2$ =				68.6400	34.32000	
$12036x/10$ =					1203.60000	+16930

$f(x) = .00120\,{}^6T_3 + .0840\,{}^5T_3 + 2.9823\,{}^4T_2 + 68.6470\,{}^3T_2 + 1238.41351\,{}^2T_1 + 16930\,{}^1T_0$ provides the coefficients for subtabulating the function at the desired interval, beginning at the argument $x = 7.0$.

7. Accuracy of Tables. When the values of the coefficients are not exact, owing to the original values for a, b, c etc. or the dropping of decimals in the computation of the coefficients, the errors accumulate fairly rapidly. Each coefficient will introduce its own error into the summation.

To maintain accuracy throughout a long table it is advisable to transform $f(x)$ by Horner's method of decreasing the roots [10, pp. 100–101], compute new coefficients for the transformed equation at intervals, and prepare the table in sections. Decreasing the roots by r gives us a new starting point at $x = r$.

Since two or more functions may be computed at one time, a function for which the coefficients are not exact may be computed by adding in the usual way from the starting values and subtracting from the ending values simultaneously. As many digits as agree in both tabulations of the function may be considered correct.

The tabulations can be made to practically any degree of accuracy on the equipment available, as the newer machines can be formed into counters of any capacity up to 80 digits. In practice, counters of 16, 20 or 24 digits will ordinarily suffice for the accuracy desired and two or more functions can be evaluated simultaneously. Cards are read and added at the rate of 150 per minute, or read, added and listed on the tape at the rate of 80 per minute and new summary cards produced at the rate of 40 per minute (on alphabetic equipment with gang summary punches). Computation may be carried out with additional decimal places and the final tabulation of the function rounded off to the nearest number retained.

8. Summary. The cumulative or progressive-total method is shown to be applicable to the preparation of tables of functions expressed in the form of a power series.

The cumulative formulas for the powers through the twelfth power have been presented, and simple methods are given for transforming a power series into its corresponding cumulative formula, for changing the interval of the table, rounding off the values of the function, and subdividing the table at desired points.

It is hoped that this discussion will make tables in printed or punched-card form more generally available as a tool for the computer. Since tables may be so readily prepared by this process, the usefulness of the tabular method of solving problems is greatly increased.

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REFERENCES

[1] G. F. Hardy, *Theory of Construction of Tables of Mortality*, pp. 59–62, 124–128.

[2] R. M. Mendenhall and R. Warren, *The Mendenhall-Warren-Hollerith Correlation Method*, Columbia University Statistical Bureau Document, No. 1, 1929, Columbia University, New York.

[3] R. M. Mendenhall and R. Warren, "Computing statistical coefficients from punched cards," *Jour. Educ. Psych.*, Vol. 21 (1930), pp. 53–62.

[4] H. S. Razram and M. E. Wagner, "The summation method in statistics," *Jour. Exper. Psych.*, Vol. 14 (1931), pp. 270–283.

[5] A. E. Brandt, *Uses of the Progressive Digit Method*, Punched Card Method in Colleges and Universities, pp. 423–436.

[6] P. S. DWYER, "The computation of moments with the use of cumulative totals," *Annals of Math. Stat.*, Vol. 9 (1938), pp. 288–304.

[7] P. S. DWYER, "The cumulative numbers and their polynomials," *Annals of Math. Stat.*, Vol. 9 (1940), pp. 66–71.

[8] L. J. COMRIE, *Interpolation and Allied Tables*, Reprinted from the Nautical Almanac for 1937, His Majesty's Stationery Office.

[9] E. C. BOWER, "Systematic subdivision of tables," Lick Observatory Bulletin 467 (1935), University of California Press.

[10] WHITTAKER and ROBINSON, *The Calculus of Observations*, Blackie and Son, Ltd., London, 1932.

ON THE PROBABILITY OF THE OCCURRENCE OF AT LEAST m EVENTS AMONG n ARBITRARY EVENTS

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Introduction. Let $E_1, \cdots, E_n$, denote n arbitrary events. Let $p_{\nu_1\cdots\nu_i\nu_{i+1}\cdots\nu_j}$, where $0 \leqq i \leqq j \leqq n$ and $(\nu_1, \cdots, \nu_j)$ is a combination of the integers $(1, \cdots, n)$, denote the probability of the non-occurrence of $E_{\nu_1}, \cdots, E_{\nu_i}$ and the occurrence of $E_{\nu_{i+1}}, \cdots, E_{\nu_j}$. Let $p_{[\nu_1\cdots\nu_i]}$ denote the probability of the occurrence of $E_{\nu_1}, \cdots, E_{\nu_i}$ and no others among the n events. Let $S_j = \Sigma p_{\nu_1\cdots\nu_j}$ where the summation extends to all combinations of j of the n integers $(1, \cdots, n)$. Let $p_m(\nu_1, \cdots, \nu_k)$, $(1 \leqq m \leqq k \leqq n)$, denote the probability of the occurrence of at least m events among the k events $E_{\nu_1}, \cdots, E_{\nu_k}$.

By the set $(x_1, \cdots, x_b, \cdots, x_a) - (x_1, \cdots, x_b)$ (where $b \leqq a$) we mean the set $(x_{b+1}, \cdots, x_a)$. And by a $\binom{a}{b}$-combination out of $(x_1, \cdots, x_a)$ we mean a combination of b integers out of the a integers $(x_1, \cdots, x_a)$.

We often use summation signs with their meaning understood, thus for a fixed k, $1 \leqq k \leqq n$, the summations in $\Sigma p_{\nu_1\cdots\nu_k}$, or $\Sigma p_m(\nu_1, \cdots, \nu_k)$, extend to all the $\binom{n}{k}$-combinations out of $(1, \cdots, n)$.

The following conventions concerning the binomial coefficients are made:

$$\binom{0}{0} = 1, \qquad \binom{a}{b} = 0 \quad \text{if} \quad a < b \quad \text{or if} \quad b < 0.$$

It is a fundamental theorem in the theory of probability that, if $E_1, \cdots, E_n$ are incompatible (or "mutually exclusive"), then

$$p_1(1, \cdots, n) = p_1 + \cdots + p_n.$$

When the events are arbitrary, we have Boole's inequality

$$p_1(1, \cdots, n) \leqq p_1 + \cdots + p_n.$$

Gumbel[1] has generalized this inequality to the following:

$$p_1(1, \cdots, n) \leqq \frac{\Sigma p_1(\nu_1, \cdots, \nu_k)}{\binom{n-1}{k-1}},$$

[1] *C. R. Acad. Sc.* Vol. 205(1937), p. 774.

for $k = 1, \cdots, n$. The case $k = 1$ gives Boole's inequality. Fréchet[2] has announced that Gumbel's result can be sharpened to the following

$$A_{k+1} = \frac{\Sigma p_1(\nu_1, \cdots, \nu_{k+1})}{\binom{n-1}{k}} \leqq \frac{\Sigma p_1(\nu_1, \cdots, \nu_k)}{\binom{n-1}{k-1}} = A_k, \tag{1}$$

for $k = 1, \cdots, n-1$. Thus, A_k is non-increasing for k increasing. On the other hand, Poincaré has obtained the following formula which expresses $p_1(1, \cdots, n)$ in terms of the S_j's,

$$\begin{aligned} p_1(1, \cdots, n) &= \sum p_{\nu_1} - \sum p_{\nu_1\nu_2} + \sum p_{\nu_1\nu_2\nu_3} - \cdots \\ &\quad + (-1)^n p_{1\cdots n} = \sum_{j=1}^{n} (-1)^{j-1} S_j. \end{aligned} \tag{2}$$

In the present paper we shall study the more general function $p_m(\nu_1, \cdots, \nu_k)$ as defined above. First we generalize Poincaré's formula and Fréchet's inequalities. In Theorem 1 we establish (for $1 \leqq m \leqq n$)

$$\begin{aligned} p_m(1, \cdots, n) &= \sum p_{\nu_1\cdots\nu_m} - \binom{m}{1} \sum p_{\nu_1\cdots\nu_{m+1}} \\ &\quad + \binom{m+1}{2} \sum p_{\nu_1\cdots\nu_{m+2}} + \cdots + (-1)^{n-m} \binom{n-1}{m-1} p_{1\cdots n} \\ &= \sum_{i=0}^{n-m} (-1)^i \binom{m+i-1}{i} S_{m+i}. \end{aligned} \tag{3}$$

Although this result is well known, we prove it in preparation for Theorem 2. Theorem 3 establishes

$$A_{k+1}^{(m)} = \frac{\Sigma p_m(\nu_1, \cdots, \nu_{k+1})}{\binom{n-m}{k+1-m}} \leqq \frac{\Sigma p_m(\nu_1, \cdots, \nu_k)}{\binom{n-m}{k-m}} = A_k^{(m)}, \tag{4}$$

for $k = 1, \cdots, n-1$ and $1 \leqq m \leqq k$.

Next, we extend the inequalities (4), and in Theorem 4 we show that

$$A_k^{(m)} \leqq \tfrac{1}{2}(A_{k-1}^{(m)} + A_{k+1}^{(m)}); \tag{5}$$

which states that the differences $A_k - A_{k+1}$ $(k = 1, \cdots, n-1)$ are non-decreasing for increasing k. From this and a simple result we can deduce (4). Also Theorem 2 establishes that

$$\sum_{i=0}^{2l+1} (-1)^i \binom{m+i-1}{i} S_{m+i} \leqq p_m(1, \cdots, n) \leqq \sum_{i=0}^{2l} (-1)^i \binom{m+i-1}{i} S_{m+i}, \tag{6}$$

[2] Loc. cit., Vol. 208(1939), p. 1703.

for $2l + 1 \leqq n - m$ and $2l \leqq n - m$ respectively. These inequalities throw light on formula (3) and are sharper than the following analogue of Boole's inequality for $p_m(1, \cdots, n)$, which is a special case of (4):

$$p_m(1, \cdots, n) \leqq \Sigma p_{\nu_1 \cdots \nu_m} . \tag{7}$$

The last statement will be evident in the proof.

In Theorem 5 we give an "inversion" of the formula (3), i.e. we express $p_{1\cdots n}$ in terms of the $p_m(\nu_1, \cdots, \nu_k)$'s, as follows:

$$\begin{aligned} \binom{n-1}{m-1} p_{1\cdots n} &= \sum p_m(\nu_1, \cdots, \nu_m) - \sum p_m(\nu_1, \cdots, \nu_{m+1}) + \cdots \\ &\qquad + (-1)^{n-m} p_m(1, \cdots, n) \\ &= \sum_{i=0}^{n-m} (-1)^i \sum p_m(\nu_1, \cdots, \nu_{m+i}). \end{aligned} \tag{8}$$

This of course implies the following more general formula for $p_{\alpha_1 \cdots \alpha_r}$,

$$\binom{r-1}{m-1} p_{\alpha_1 \cdots \alpha_r} = \sum_{i=0}^{r-m} (-1)^i \sum p_m(\nu_1, \cdots, \nu_{m+i})$$

where $(\alpha_1, \cdots, \alpha_r)$ is a combination of the integers $(1, \cdots, n)$ and where the second summation extends to all the $\binom{r}{m+i}$-combinations of $(\alpha_1, \cdots, \alpha_r)$. Since it is known[3] that we can express other functions such as S_r, $p_{[\mu_1 \cdots \mu_r]}$ in terms of the $p_{\mu_1 \cdots \mu_r}$'s, we can also express them in terms of the $p_m(\nu_1, \cdots, \nu_k)$'s, provided $r \geqq m$.

Finally, for the case $m = 1$, we give in Theorem 6 an explicit formula for $p_{[1\cdots r]}$ in terms of the $p_1(\nu_1, \cdots, \nu_k)$'s, as shown in (9),

$$\begin{aligned} p_{[1\cdots r]} &= - p_1(r+1, \cdots, n) + \sum_{\nu_1} p_1(\nu_1, r+1, \cdots, n) \\ &\qquad - \sum_{\nu_1, \nu_2} p_1(\nu_1, \nu_2, r+1, \cdots, n) + \cdots \\ &\qquad + (-1)^{r-1} \sum p_1(1, \cdots, r, r+1, \cdots, n), \\ &= \sum_{i=1}^{r} (-1)^{i-1} \sum_{(\nu_1, \cdots, \nu_i)} p_1(\nu_1, \cdots, \nu_i, r+1, \cdots, n), \end{aligned} \tag{9}$$

where $(\nu_1, \cdots, \nu_i)$ runs through all the $\binom{r}{i}$-combinations from $(1, \cdots, r)$. This of course implies the following more general formula:

$$p_{[\alpha_1 \cdots \alpha_r]} = \sum_{i=1}^{r} (-1)^{i-1} \sum_{(\nu_1, \cdots, \nu_i)} p_1(\nu_1, \cdots, \nu_i, \alpha_{r+1}, \cdots, \alpha_n),$$

[3,4] Fréchet, "Condition d'existence de systemes d'événements associés à certaines probabilités," *Jour. de Math.*, (1940), p. 51-62.

where $(\alpha_1, \cdots, \alpha_r, \cdots \alpha_n)$ is a permutation of $(1, \cdots, n)$ and where $(\nu_1, \cdots, \nu_i)$ runs through all the $\binom{r}{i}$-combinations out of $(\alpha_1, \cdots, \alpha_r)$. From Theorem 6 and two lemmas we deduce a condition of existence of systems of events associated with the probabilities $p_1(\nu_1, \cdots, \nu_m)$. The author has not been able to obtain similar elegant results for the general m. Probably they do not exist.

2. Generalization of Poincaré's formula; Generalization and sharpening of Boole's inequality.

THEOREM 1:

$$p_m(1, \cdots, n) = \sum p_{\nu_1 \cdots \nu_m} - \binom{m}{1} \sum p_{\nu_1 \cdots \nu_{m+1}} + \binom{m+1}{2} \sum p_{\nu_1 \cdots \nu_{m+2}} - \cdots + (-1)^{n-m} \binom{n-1}{n-m} p_{1 \cdots n}. \tag{3}$$

PROOF: We have

$$p_m(1, \cdots, n) = \sum_{b=0}^{n-m} \sum p_{[\mu_1 \cdots \mu_{m+b}]}, \tag{10}$$

where the second summation extends, for a fixed b, to all the $\binom{n}{m+b}$-combinations of $(1, \cdots, n)$. Further we have

$$p_{\nu_1 \cdots \nu_{m+c}} = \sum_{d=0}^{n-m-c} \sum p_{[\nu_1 \cdots \nu_{m+c} \cdots \nu_{m+c+d}]} \tag{11}$$

where the second summation extends, for a fixed d, to all the $\binom{n-m-c}{d}$-combinations of $(1, \cdots, n) - (\nu_1, \cdots, \nu_{m+c})$. The formulas (10) and (11) are evident by observing that the probabilities in the summations are all additive. Now we count the number of times a fixed $p_{[\mu_1 \cdots \mu_{m+b}]}$ appears in (3). By (11) this is equal to the sum

$$\binom{m+b}{m} - \binom{m}{1}\binom{m+b}{m+1} + \binom{m+1}{2}\binom{m+b}{m+2} - \cdots + (-1)^{n-m}\binom{n-1}{n-m}\binom{m+b}{m+b} = 1,$$

since this number is the coefficient of $(-1)^m x^m$ in the expansion of

$$(1-x)^{m+b}\left(1 - \frac{1}{x}\right)^{-m} = (-1)^{-m} x^m (1-x)^b.$$

Thus by (10) we have (3).

THEOREM 2: *For $2l \leqq n - m$ and $2l \leqq n - m$ respectively, we have*

$$(6)\quad \sum_{i=0}^{2l+1} (-1)^i \binom{m+i-1}{i} S_{m+i} \leqq p_m(1, \cdots, n) \leqq \sum_{i=0}^{2l} (-1)^i \binom{m+i-1}{i} S_{m+i}.$$

PROOF: By the reasoning in the previous proof, it is sufficient (in fact also necessary) to show that

$$\sum_{i=0}^{2l} \binom{m-1+i}{i}\binom{m+b}{m+i} \geqq 1, \qquad \sum_{i=0}^{2l+1} \binom{m-1+i}{i}\binom{m+b}{m+i} < 1.$$

Since

$$\binom{m-1+i}{i}\binom{m+b}{m+i} = \frac{(m+b)!}{(m-1)!\,b!}\binom{b}{i}\frac{1}{m+i}$$

is an integer, it is sufficient to show that

$$(12)\qquad \sum_{i=0}^{2l} (-1)^i \binom{b}{i}\frac{1}{m+i} > 0, \qquad \sum_{i=0}^{2l+1} (-1)^i \binom{b}{i}\frac{1}{m+i} \leqq 0.$$

Suppose $b > 0$ is even. For $i \leqq b/2 - 1$, we have $\dfrac{b-i}{i+1} > 1$ so that $\dfrac{b-i}{i+1} \geqq \dfrac{i+2}{i+1}$. Also $\dfrac{m+i}{m+i+1} \geqq \dfrac{i+1}{i+2}$ for $m \geqq 1$. Hence

$$\binom{b}{i+1}\frac{1}{m+i+1} = \frac{b-i}{i+1}\frac{m+i}{m+i+1}\binom{b}{i}\frac{1}{m+i}$$

$$\geqq \frac{i+2}{i+1}\frac{i+1}{i+2}\binom{b}{i}\frac{1}{m+i} = \binom{b}{i}\frac{1}{m+i}.$$

For $i \geqq b/2$ we have $\dfrac{b-i}{i+1} < 1$ so that $\dfrac{b-i}{i+1}\dfrac{m+i}{m+i+1} < 1$ and

$$\binom{b}{i+1}\frac{1}{m+i+1} < \binom{b}{i}\frac{1}{m+i}.$$

Thus the absolute values of the terms of the alternating series

$$\sum_{i=0}^{b} (-1)^i \binom{b}{i}\frac{1}{m+i} = \frac{b!}{(m+b)!(m-1)!}$$

are monotone increasing as long as $i \leqq \dfrac{b}{2} - 1$, reaching maximum at $i = \dfrac{b}{2}$ and then become monotone decreasing.

Therefore (12) evidently holds for $2l \leqq b/2$ and $2l + 1 \leqq b/2$ respectively. For $t \geqq \dfrac{b}{2} + 1$ we write

$$\sum_{i=0}^{t} (-1)^i \binom{b}{i}\frac{1}{m+i} = \frac{b!}{(m+b)!(m-1)!} - \sum_{i=t+1}^{b} (-1)^i \binom{b}{i}\frac{1}{m+i}$$

$$= \frac{b!}{(m+b)!(m-1)!} - \sum_{j=0}^{b-t-1} (-1)^j \binom{b}{j}\frac{1}{m+b-j}.$$

From the above and the fact that $\frac{b!}{(m+b)!(m-1)!} \leqq \frac{1}{m+b}$ we see that the righthand side is an alternating series whose terms are non-decreasing in absolute values. Hence (12) is true.

If b is odd, the case is similar.

3. Generalization of Fréchet's inequalities and related inequalities. Before proving our remaining theorems, we shall give a more detailed account of the general method which will be used. In the foregoing work we have already given two different expressions for the function $p_m(1, \cdots, n)$, namely, formulas (3) and (10), but they are not convenient for our later purposes. Formula (3) is inconvenient because it is not additive and because the $p_{\nu_1 \cdots \nu_i}$'s are related in magnitudes; while formula (10) has gone so far in the separation of the additive constituents that its application raises algebraical difficulties. Let us therefore take an intermediate course.

Let each $\binom{n}{m}$-combination $(\nu_1, \cdots, \nu_m)$ out of $(1, \cdots, n)$ be written so that $\nu_1 < \nu_2 < \cdots < \nu_m$. Then we arrange them in an ordered sequence in the following way: the combination $(\nu_1, \cdots, \nu_m)$ is to precede the combination $(\mu_1, \cdots, \mu_m)$ if, for the first $\nu_i \neq \mu_i$, we have $\nu_i > \mu_i$. After such an arrangement we symbolically denote these combinations by

$$\text{I, II, } \cdots, \left[\binom{n}{m}\right].$$

Further, all the $\binom{k}{m}$-combinations out of $(\nu_1, \cdots, \nu_k)$ where the latter is a combination out of $(1, \cdots, n)$ are arranged in the order in which they appear in the sequence just written. For example, all the $\binom{4}{2}$-combinations out of (1, 2, 3, 4) are ordered thus:

$$(12) \quad (13) \quad (14) \quad (23) \quad (24) \quad (34).$$

Let U denote a typical combination $(\mu_1, \cdots, \mu_m)$. By E_U we mean the combination of events $E_{\mu_1}, \cdots, E_{\mu_m}$ so that $p_U = p_{\mu_1 \cdots \mu_m}$. In general, let the combinations $U_1, \cdots, U_{b-1}, U_b$ be given, then $p_{U_1' \cdots U_{b-1}' U_b}$ denotes the probability of the non-occurrence of $U_1, \cdots, U_{b-1}$ and the occurrence of U_b.

Now let $\text{I, II, } \cdots, \left[\binom{k}{m} - 1\right] = Y, \left[\binom{k}{m}\right] = Z$ denote all the $\binom{k}{m}$-combinations out of $(\nu_1, \cdots, \nu_k)$ in their assigned order. We have

$$p_m(\nu_1, \cdots, \nu_k) = p_{\text{I}} + p_{\text{I}'\text{II}} + p_{\text{I}'\text{II}'\text{III}} + \cdots + p_{\text{I}' \cdots \text{Y}'\text{Z}}. \tag{13}$$

This fundamental formula is evident. Of course it is possible to identify the p's on the right-hand side with the ordinary $p_{\nu_1' \cdots \nu_j'}$'s, but we shall refrain from so doing and be content with the following example:

$$p_2(1, 2, 3, 4) = p_{12} + p_{12'3} + p_{12'3'4} + p_{1'23} + p_{1'23'4} + p_{1'2'34}.$$

THEOREM 3. *For $k = 1, \cdots, n-1$ and $1 \leqq m \leqq k$ we have*

$$\binom{n-m}{k-m} \Sigma p_m(\nu_1, \cdots, \nu_{k+1}) \leqq \binom{n-m}{k+1-m} \Sigma p_m(\nu_1, \cdots, \nu_k).$$

PROOF. Substitute (13) and a similar formula for $k+1$ into the two sides respectively. After this substitution we observe that the number of terms is the same on both sides, since

$$\binom{n-m}{k-m}\binom{n}{k+1}\binom{k+1}{m} = \binom{n-m}{k+1-m}\binom{n}{k}\binom{k}{m}.$$

Also, the number of terms with a given $U = (\mu_1, \cdots, \mu_m)$ unaccented is the same, since

$$\binom{n-m}{k-m}\binom{n-m}{k+1-m} = \binom{n-m}{k+1-m}\binom{n-m}{k-m}.$$

Let the sum of all the terms with U unaccented in the two summations be denoted by $\sigma_{k+1} = \sigma_{k+1}(\mu_1, \cdots, \mu_m)$ and $\sigma_k = \sigma_k(\mu_1, \cdots, \mu_m)$ respectively. It is sufficient to prove that

$$\binom{n-m}{k-m}\sigma_{k+1} \leqq \binom{n-m}{k+1-m}\sigma_k, \tag{14}$$

for any U. σ_k contains $\binom{n-m}{k-m}$ terms each of the form $p_{\nu_1'\cdots\nu_l'\mu_1\cdots\mu_m}$ where $0 \leqq l \leqq \mu_m - m$ and where $(\nu_1, \cdots, \nu_l, \mu_1, \cdots, \mu_m)$ is a $\binom{\mu_m}{m+l}$-combination out of $(1, \cdots, \mu_m)$. For fixed $(\mu_1, \cdots, \mu_m)$ and a fixed l but varying λ's, σ_k contains $\binom{n-\mu_m}{k-m-l}$ terms of the form $p_{\nu_1'\cdots\nu_l'\mu_1\cdots\mu_m}$, with exactly l accented subscripts. Let the sum of all such terms be denoted by $\sigma_k^{(l)}$. Evidently $\sigma_k^{(l)}$ has $\binom{\mu_m - m}{l}$ terms. As a check we have

$$\binom{n-\mu_m}{k-m}\binom{\mu_m - m}{0} + \binom{n-\mu_m}{k-m-1}\binom{\mu_m - m}{1} + \cdots + \binom{n-\mu_m}{k-\mu_m}\binom{\mu_m - m}{\mu_m - m} = \binom{n-m}{k-m},$$

which is the total number of terms in σ_k.

We decompose these p's partially, as follows:

$$p_{\nu_1'\cdots\nu_l'\mu_1\cdots\mu_m} = \sum_{b=0}^{\mu_m - m - l} \sum_{\mu_{m+1}\cdots\mu_{m+b}} p_{\nu_1'\cdots\nu_{l+c}'\mu_1\cdots\mu_{m+b}},$$

where $(\nu_1, \cdots, \nu_{l+c}, \mu_1, \cdots, \mu_{m+b})$ is a permutation of $(1, \cdots, \mu_m)$ and where the second summation extends, for a fixed b, to all the $\binom{\mu_m - m - l}{b}$-combinations out of $(1, \cdots, \mu_m) - (\nu_1, \cdots, \nu_l, \mu_1, \cdots \mu_m)$.

Now consider a given

$$p_{\rho_1 \cdots \rho_t \lambda_1 \cdots \lambda_s \mu_1 \cdots \mu_m}$$

where $0 \leqq t \leqq \mu_m - m$ and $(\rho_1 \cdots \rho_t \lambda_1 \cdots \lambda_s \mu_1 \cdots \mu_m)$ is a permutation of $(1, \cdots, \mu_m)$. It appears $\binom{t}{l}$ times in $\sigma_k^{(l)}$. Hence it appears

$$\binom{n-\mu_m}{k-m}\binom{t}{0} + \binom{n-\mu_m}{k-m-1}\binom{t}{1} + \cdots + \binom{n-\mu_m}{k-m-t}\binom{t}{t} = \binom{n-\mu_m+t}{k-m}$$

times in σ_k.

Therefore to prove (14) it is sufficient to prove that

$$\binom{n-m}{k-m}\binom{n-\mu_m+t}{k+1-m} \leqq \binom{n-m}{k+1-m}\binom{n-\mu_m+t}{k-m}.$$

By an easy reduction we have

$$(n - \mu_m + t - k + m) \leqq n - k$$

or

$$-\mu_m + t + m \leqq 0;$$

since $t \leqq \mu_m - m$ this is obvious.

THEOREM 4: *For $2 \leqq k \leqq n-1$ and $1 \leqq m \leqq k$ we have*

$$(5) \qquad \frac{\Sigma p_m(\nu_1, \cdots, \nu_k)}{\binom{n-m}{k-m}} \leqq \frac{1}{2}\frac{\Sigma p_m(\nu_1, \cdots, \nu_{k-1})}{\binom{n-m}{k-1-m}} + \frac{1}{2}\frac{\Sigma p_m(\nu_1, \cdots, \nu_{k+1})}{\binom{n-m}{k+1-m}}.$$

PROOF: By the reasoning in the previous proof, it is sufficient to show that

$$2\binom{n-m}{k-1-m}\binom{n-m}{k+1-m}\binom{n-\mu_m+t}{k-m} \leqq \binom{n-m}{k-m}\binom{n-m}{k+1-m}\binom{n-\mu_m+t}{k-1-m} + \binom{n-m}{k-m}\binom{n-m}{k-1-m}\binom{n-\mu_m+t}{k+1-m},$$

for $0 \leqq t \leqq \mu_m - m$. By an easy reduction this is equivalent to

$$2(n-k)(n-\mu_m+t-k+m+1) \leqq (n-k+1)(n-k) + (n-\mu_m+t-k+m+1)(n-\mu_m+t-k+m)$$

or

$$(n-\mu_m+t-k+m+1)(\mu_m-t-m) \leqq (n-k)(\mu_m-t-m).$$

For $t = \mu_m - m$ we have equality, otherwise we have

$$-\mu_m + t + m + 1 \leqq 0.$$

We can deduce Theorem 3 from Theorem 4 and the following result (a case of generalized Gumbel inequalities):

$$\binom{n-1}{n-2} p_m(1, \cdots, n) \leqq \Sigma p_m(\nu_1, \cdots, \nu_{n-1}). \tag{15}$$

PROOF OF (15): Substitute from (13). Consider the p's with U unaccented. The number of such terms is the same on both sides. But on the left-hand side they are all the same $p_{I'II'\cdots(U-1)'U}$, while those on the right-hand side, being of the form $p_{U_1'\cdots U_\lambda' U}$ where $0 \leqq \lambda \leqq U-1$ and $(U_1, \cdots, U_\lambda)$ is a combination out of $(1, \cdots, U-1)$, are greater than or equal to it. Hence the result.

4. The $p_{\alpha_1\cdots\alpha_i}$'s in terms of the $p_m(\nu_1, \cdots, \nu_k)$'s and the $p_{[\alpha_1\cdots\alpha_i]}$'s in terms of the $p_1(\nu_1, \cdots, \nu_k)$'s.

THEOREM 5: *For $1 \leqq m \leqq n$ we have*

$$\begin{aligned}\binom{n-1}{m-1} p_{1\cdots n} &= \sum p_m(\nu_1, \cdots, \nu_m) - \sum p_m(\nu_1, \cdots, \nu_{m+1}) + \cdots \\ &\qquad + (-1)^{n-m} p_m(1, \cdots, n) \\ &= \sum_{i=0}^{n-m} (-1)^i \sum_{\nu_1, \cdots \nu_{m+i}} p_m(\nu_1, \cdots, \nu_{m+i}).\end{aligned} \tag{8}$$

PROOF: As in the proof of Theorem 3, consider $\sigma_k(\mu_1, \cdots, \mu_m)$. Here $m \leqq k \leqq n$. Since a given

$$p_{\rho_1'\cdots\rho_t'\lambda_1\cdots\lambda_s\mu_1\cdots\mu_m}, \tag{16}$$

appears $\binom{n-\mu_m+t}{k-m}$ times in σ_k, it appears

$$\begin{aligned}\sum_{k=m}^{n} (-1)^{k-m}\binom{n-\mu_m+t}{k-m} &= \sum_{j=0}^{n-m} (-1)^j \binom{n-\mu_m+t}{j} \\ &= \sum_{j=0}^{n-\mu_m+t} (-1)^j \binom{n-\mu_m+t}{j} = \begin{cases} 0, & \text{if } n-\mu_m+t \geqq 1, \\ 1, & \text{if } n-\mu_m+t = 0. \end{cases}\end{aligned}$$

times on the right hand side of (8). Hence for fixed $(\mu_1, \cdots, \mu_m)$, the only p's of the form (16) which actually appears are those with $t = \mu_m - n$. But $\mu_m \leqq n$, thus $t = 0$, $\mu_m = n$, and $(\lambda_1, \cdots \lambda_s, \mu_1, \cdots, \mu_m)$ is a permutation of $(1, \cdots, n)$. The term in question is therefore $p_{1\cdots n}$. Since the number of $\binom{n}{m}$-combinations of $(1, \cdots, n)$ with $\mu_m = n$ is $\binom{n-1}{m-1}$, we have the theorem.

THEOREM 6: *For $1 \leqq r \leqq n-1$, we have*

$$\begin{aligned}p_{[1\cdots r]} &= -p_1(r+1, \cdots, n) + \sum_{\nu_1} p_1(\nu_1, r+1, \cdots, n) \\ &\quad - \sum_{\nu_1,\nu_2} p(\nu_1, \nu_2, r+1, \cdots, n) + \cdots + (-1)^{r-1} \sum p_1(1, \cdots, n) \\ &= \sum_{i=1}^{r} (-1)^{i-1} \sum_{\nu_1, \cdots, \nu_i} p_1(\nu_1, \cdots, \nu_i, r+1, \cdots, n),\end{aligned} \tag{9}$$

where $(\nu_1, \cdots, \nu_i)$ *runs through all the* $\binom{r}{i}$*-combinations out of* $(1, \cdots, r)$.

PROOF: We rewrite (14) for the special case $m = 1$,

$$p_1(\mu_1, \cdots, \mu_k) = p_{\mu_1} + p_{\mu_1'\mu_2} + \cdots + p_{\mu_1'\cdots\mu_{k-1}'\mu_k}, \tag{17}$$

where $\mu_1 < \mu_2 < \cdots < \mu_k$. Substitute into the right hand side of (9). After the substitution let the sum of all those p's with μ unaccented be denoted by σ_μ. The terms in σ_μ are of the form $p_{\mu_1'\cdots\mu_{s-1}'\mu}$ where $1 \leqq s \leqq \mu$ and $(\mu_1, \cdots, \mu_{s-1})$ is a combination out of $(1, \cdots, \mu - 1)$.

First consider a fixed $\mu \leqq r$. For a fixed $p_{\mu_1'\cdots\mu_{s-1}'\mu}$ we count the number of times it appears in σ_μ, that is, on the right hand side of (9). This is evidently equal to

$$\sum_{j=s}^{r} (-1)^j \binom{r-\mu}{j-s} = \sum_{j=s}^{r-\mu+s} (-1)^j \binom{r-\mu}{j-s} = \begin{cases} 0, & \text{if } r - \mu \geqq 1, \\ 1, & \text{if } r - \mu = 0. \end{cases}$$

Thus the only terms that actually appear are those with $\mu = r$; and each of such terms $p_{\mu_1'\cdots\mu_{s-1}'r}$ appears exactly once with the sign $(-1)^s$. Hence their total contribution is

$$p_r - \sum_{\nu_1} p_{\nu_1' r} + \sum_{\nu_1, \nu_2} p_{\nu_1'\nu_2' r} - \cdots + (-1)^{r-1} p_{1'\cdots(r-1)'r} = p_{1\cdots r}, \tag{18}$$

by an easy modification of Poincaré's formula.

Next consider a fixed $\mu \geqq r + 1$. Every term with μ unaccented in σ_μ is of the form (with the usual convention for $\mu = r + 1$) $p_{\mu_1'\cdots\mu_s'(r+1)'\cdots(\mu-1)'\mu}$, where $(\mu_1, \cdots, \mu_s)$ is a combination out of $(1, \cdots, r)$; and it appears exactly once with the sign $(-1)^s$. Their total contribution is therefore

$$- p_{(r+1)'\cdots(\mu-1)'\mu} + \sum_{\nu_1} p_{\nu_1'(r+1)'\cdots(\mu-1)'\mu} - \sum_{\nu_1, \nu_2} p_{\nu_1'\nu_2'(r+1)'\cdots(\mu-1)'\mu} + \cdots$$
$$+ (-1)^{r-1} p_{1'\cdots(\mu+1)'\mu} = - p_{1\cdots r(r+1)'\cdots(\mu-1)'\mu},$$

by another application of Poincaré's formula. Summing up for $\mu = r + 1, \cdots, n$, we obtain

$$-(p_{1\cdots r(r+1)} + p_{1\cdots r(r+1)'(r+2)} + \cdots + p_{1\cdots r(r+1)'\cdots(n-1)'n}). \tag{19}$$

Adding (18) and (19), we obtain as the sum of the right-hand side of (9)

$$p_{1\cdots r} - (p_{1\cdots r(r+1)} + p_{1\cdots r(r+1)'(r+2)} + \cdots + p_{1\cdots r(r+1)'\cdots(n-1)'n})$$
$$= p_{1\cdots r(r+1)'(r+2)'\cdots n'} = p_{[1\cdots r]}$$

by an easy modification of (17).

5. A condition for existence of systems of events associated with the probabilities $p_1(\nu_1, \cdots, \nu_k)$.

LEMMA 1: *Let any* $2^n - 1$ *quantities* $q(\alpha_1, \cdots, \alpha_k)$ *be given, where* $k =$

$1, \cdots, n$, *and for a fixed* k, $(\alpha_1, \cdots, \alpha_k)$ *runs through all the* $\binom{n}{k}$*-combinations out of* $(1, \cdots, n)$. *Let the quantities* $Q(\alpha_1, \cdots, \alpha_k)$ *be formed as follows*:

$$Q(0) = 1 - q(1, \cdots, n),$$

$$Q(\alpha_1, \cdots, \alpha_k) = -q(\alpha_{k+1}, \cdots, \alpha_n) + \sum_{\nu_1} q(\nu_1, \alpha_{k+1}, \cdots, \alpha_n) - \sum_{\nu_1, \nu_2} q(\nu_1, \nu_2, \alpha_{k+1}, \cdots, \alpha_n) + \cdots + (-1)^{k-1} q(1, \cdots, n),$$

where $(\nu_1, \cdots, \nu_i)$ runs through all the $\binom{k}{i}$-combinations out of $(1, \cdots, n) - (\alpha_{k+1}, \cdots, \alpha_n)$. *Then the sum of all these* Q*'s is equal to* 1.

PROOF: Add all these Q's and count the number of times a fixed $q(\mu_1, \cdots, \mu_k)$ appears in the sum. For $1 \leqq k \leqq n$ this number is equal to

$$-1 + \binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k-1}\binom{k}{k} = 0.$$

Hence we have the lemma.

LEMMA 2: (Fréchet) *Given* 2^n *quantities* $Q_{[\alpha_1 \cdots \alpha_r]}$ *where* $(\alpha_1, \cdots, \alpha_r)$ *runs through all combinations out of* $(1, \cdots, n)$ *including the empty one. The necessary and sufficient condition that there exist systems of events* $E_1, \cdots, E_n$ *for which*

$$p_{[\alpha_1 \cdots \alpha_r]} = Q_{[\alpha_1 \cdots \alpha_r]}$$

(where $p_{[0]}$ denotes the probability for the non-occurrence of $E_1, \cdots, E_n$) *is that each* $Q \geqq 0$ *and that their sum is equal to* 1.

PROOF: Since the probabilities $p_{[\alpha_1 \cdots \alpha_r]}$ are independent, i.e., unrelated in magnitudes except that their sum is equal to 1, the lemma is evident.

THEOREM 7: *Given* $2^n - 1$ *quantities* $q(\alpha_1, \cdots, \alpha_k)$ *as in* Lemma 1, *the necessary and sufficient condition that there exist systems of events* $E_1, \cdots, E_n$ *for which*

$$p_1(\alpha_1, \cdots, \alpha_k) = q(\alpha_1, \cdots, \alpha_k)$$

is that for any combination $(\alpha_{r+1}, \cdots, \alpha_n)$, $1 \leqq r \leqq n - 1$, *out of* $(1, \cdots, n)$ *we have*

$$-q(\alpha_{r+1}, \cdots, \alpha_n) + \sum_{\nu_1} q(\alpha_{\nu_1}, \alpha_{r+1}, \cdots, \alpha_n) - \sum_{\nu_1, \nu_2} q(\alpha_{\nu_1}, \alpha_{\nu_2}, \alpha_{r+1}, \cdots, \alpha_n) + \cdots + (-1)^{r-1} q(1, \cdots, n) \geqq 0,$$

and thus

$$1 - q(1, \cdots, n) \geqq 0.$$

PROOF: The condition is necessary by Theorem 6. It is sufficient by Lemma 1, 2 and an obvious formula expressing $p_1(\alpha_1, \cdots, \alpha_r)$ in terms of the $p_{[\nu_1 \cdots \nu_i]}$'s.

NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A NOTE ON SHEPPARD'S CORRECTIONS

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As far as the author is aware, H. C. Carver was the first to point out that while the formulae ordinarily given for Sheppard's corrections for central moments are valid for moments computed about the population mean, there are still systematic errors present when they are applied to central moments calculated from any particular grouped frequency distribution [1]. This is due, of course to the fact that the mean of a grouped frequency distribution is in general different from that of the distribution before grouping. For a fixed class interval k, Sheppard's corrections give the average value of a moment about a fixed point of a given order for all the groupings of this class width possible and will fail to do so if the moment in question is calculated for each position of the class limits about a point which varies as the class limits shift. Thus Carver [1] pointed that the commonly used formula (for a continuous variate),

$$\mu_2 = \nu_2 - \frac{k^2}{12}, \tag{1}$$

should, if ν_2 is calculated about the mean of the grouped distribution as it is in practice, be replaced by

$$\mu_2 = \nu_2 - \frac{k^2}{12} + \sigma_M^2 \tag{2}$$

in which σ_M^2 is the variance of the means of grouped distributions over all positions of the class limits with the fixed class width k.

Recently J. A. Pierce [2] gave a method for deriving the required formulae of the type of (2) and gave actual formulae for both moments and seminvariants through the sixth order. It is the purpose of this note to point out that the use of moment generating functions provides a more elegant and concise way of arriving at formulae equivalent to Pierce's though in a somewhat different form. This method can be immediately extended to distributions of two or more variates.

In a previous paper [3] on Sheppard's corrections for a discrete variate, the author made use of the following argument: It is assumed that for a fixed class width k, any point in the scale on which the variate x is plotted is as likely to be

chosen as a class limit as any other; choosing a system of class limits for grouping the data is then equivalent to placing at random on the x-axis a scale with division points at intervals of k. Once the system of class limits is chosen any value of x before grouping bears to the class mark, x_i, of the class in which it falls the relation,

$$x_i = x + \epsilon, \tag{3}$$

in which x and ϵ are independent variates. The frequency law governing x, is, of course, that of the population from which it is drawn while ϵ is distributed in a rectangular distribution with the range $\left(-\frac{k}{2}, \frac{k}{2}\right)$ for a continuous variate and $\left(-\frac{m-1}{2m}k, \frac{m-1}{2m}k\right)$ if m consecutive values of a discrete variate are grouped in each class interval. In either case

$$M_{x_i}(\vartheta) = M_x(\vartheta)M_\epsilon(\vartheta) \tag{4}$$

in which $M_{x_i}(\vartheta)$ is the moment generating function of the variate x_i, etc. The expansion of both sides of (4) in powers of ϑ gives the relations between the average values of moments of the grouped distribution over all positions of the scale and the moments of the ungrouped distribution from which Sheppard's corrections are obtained by solving for the moments of the ungrouped distribution. The relations are valid for any fixed point about which the moments are computed; if this fixed point be taken as the mean of the ungrouped distribution the ordinary Sheppard's corrections for central moments result.

But it is quite easy to modify (4) to give the necessary relations in case the moments of each grouped distribution are computed about the mean of that distribution. We have only to write

$$x_i = x_i - \bar{x} + \bar{x} \tag{5}$$

in which $\bar{x}$ is the mean of the grouped distribution for which x_i is one of the class marks. Then

$$\begin{aligned} M_{x_i}(\vartheta) &= M_{x_i-\bar{x},\bar{x}}(\vartheta, \omega)\,|_{\omega=\vartheta} \\ &= M_x(\vartheta)M_\epsilon(\vartheta) \end{aligned} \tag{6}$$

If we write,

$$\lambda_{rs:k_i-\bar{x},\bar{x}} = \bar{\lambda}_{rs},$$

in which $\bar{\lambda}_{rs}$ is the product seminvariant of order rs of moments about the means of the grouped distributions and of such means, the expansion of the logarithm of the second member of (6) gives

$$1 + (\bar{\lambda}_{10} + \bar{\lambda}_{01})\vartheta + (\bar{\lambda}_{20} + 2\bar{\lambda}_{11} + \bar{\lambda}_{02})\frac{\vartheta^2}{2} + (\bar{\lambda}_{10} + \bar{\lambda}_{01})^{(3)}\frac{\vartheta^3}{3!} + \cdots, \tag{7}$$

in which

$$(\bar{\lambda}_{10} + \bar{\lambda}_{01})^{(r)} = \bar{\lambda}_{r0} + r\bar{\lambda}_{r-1,1} + \cdots + \binom{r}{k}\bar{\lambda}_{r-k,k} + \cdots + \bar{\lambda}_{0r}\,.$$

The expression of the logarithm of the right member is[3]:

$$(8) \qquad \lambda_1\vartheta + \lambda_2\frac{\vartheta^2}{2!} + \lambda_3\frac{\vartheta^3}{3!} + \cdots + \sum_{s=1}^{\infty}(-1)^{s+1}\frac{B_s k^{2s}}{2s}\left(1 - \frac{1}{m^{2s}}\right)\frac{\vartheta^{2s}}{(2s)!},$$

for a discrete variate (the result for a continuous variable is obtained merely by letting $m \to \infty$) in which λ_r is the rth seminvariant of the ungrouped distribution and B_s is the sth Bernoulli number.

We may without loss of generality take the origin for x at the mean of the ungrouped distribution so that $\lambda_1 = 0$. Further it is easy to see that

$$\bar{\lambda}_{1r} = 0, \qquad r = 0, 1, 2, 3, \cdots .$$

Consider

$$E[(x_i - \bar{x})\bar{x}^r] = \bar{\nu}_{1r}$$

For a fixed $\bar{x}$, i.e., for a given grouping, this becomes

$$\bar{x}^r E(x_i - \bar{x}) = 0$$

Then since $\bar{\nu}_{1r}$ is the average of this over all groupings with a given class interval, $\bar{\nu}_{1r} = 0$, and from the expression for $\bar{\lambda}_{1r}$ in terms of the moments $\bar{\nu}_{ij}$ it is obvious that also $\bar{\lambda}_{1r} = 0$.

Then we must also have $\bar{\lambda}_{01} = 0$ as is otherwise obvious and (7) can be rewritten

$$(9) \qquad 1 + (\bar{\lambda}_{20} + \bar{\lambda}_{02})\frac{\vartheta^2}{2} + (\bar{\lambda}_{30} + 3\bar{\lambda}_{21} + \bar{\lambda}_{03})\frac{\vartheta^3}{3!} + \cdots .$$

Now from (8) and (9) by equating coefficients of like powers of ϑ, we get the set of formulae:

$$(10) \qquad \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \bar{\lambda}_{20} + \bar{\lambda}_{02} - \left(1 - \frac{1}{m^2}\right)\frac{k^2}{12} \\ \lambda_3 &= \bar{\lambda}_{30} + 3\bar{\lambda}_{21} + \bar{\lambda}_{03} \\ \lambda_4 &= \bar{\lambda}_{40} + 4\bar{\lambda}_{31} + 6\bar{\lambda}_{22} + \bar{\lambda}_{04} + \left(1 - \frac{1}{m_4}\right)\frac{k^4}{120} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

These formulae, however, do not give the sought Sheppard's corrections for seminvariants calculated from grouped distributions of a discrete variate. See below.

Referring to formula (10), p. 58 of the author's paper cited [3], it is easily seen by comparison that the required moment formulae are obtained from the general formula

$$(11) \qquad \mu_n = \sum_{s=0}^{[n/2]}\binom{n}{2s}\alpha_{2s}(\bar{\nu}_{10} + \bar{\nu}_{01})^{(n-2s)},$$

in which α_{2s} is given by formula (9) of this former paper. For $n = 1, 2, 3, 4$ we write down immediately

$$\mu_1 = 0 \qquad (\bar{\nu}_{10} = \bar{\nu}_{01} = 0)$$

$$\mu_2 = \bar{\nu}_{20} + \bar{\nu}_{02} - \left(1 - \frac{1}{m^2}\right)\frac{k^2}{12}$$

$$\mu_3 = \nu_{30} + 3\bar{\nu}_{21} + \bar{\nu}_{03} \tag{12}$$

$$\mu_4 = \nu_{40} + 4\bar{\nu}_{31} + 6\bar{\nu}_{22} + \bar{\nu}_{04} - \left(1 - \frac{1}{m^2}\right)(\bar{\nu}_{20} + \bar{\nu}_{02})\frac{k^2}{2} + \left(1 - \frac{1}{m^2}\right)\left(7 - \frac{3}{m^2}\right)\frac{k^4}{240}.$$

In these formulae, $\bar{\nu}_{r0}$ is, of course, the average value of rth central moments about the means of grouped distributions. From the definition $\bar{\nu}_{rs} (s \neq 0)$ is the average value of the product of the rth central moment of a grouped distribution by the sth power of the mean of the same grouped distribution. Also, it must be noted that in the formulae (10) the $\bar{\lambda}_{rs}'s$ there are to be calculated by the usual formulae from the moments, $\bar{\nu}_{ij}$, and are not themselves the average values of like seminvariants calculated from the separate grouped distributions. Thus though the formulae (12) give the sought Sheppard's corrections for moments, the formulae (10) do not do the like for seminvariants in general. However, since in each grouped distribution,

$$\lambda_2 = \nu_2$$

and

$$\lambda_3 = \nu_3$$

we have, taking the expectation or average value over the grouped distributions,

$$E(\lambda_2) = E(\nu_2) = \bar{\nu}_{20} = \bar{\lambda}_{20}$$

and

$$E(\lambda_3) = E(\nu_3) = \bar{\nu}_{30} = \bar{\lambda}_{30},$$

and the first two formulae of (10) do give the Sheppard's corrections for λ_2 and λ_3 calculated from grouped distributions of a discrete variate.

But the case for λ_4 is different. In each grouped distribution,

$$\lambda_4 = \nu_4 - 3\nu_2^2,$$

and if we define l_r by

$$E(\lambda_r) = l_r,$$

we have

$$l_4 = \bar{\nu}_{40} - 3E(\nu_2^2)$$
$$= \bar{\nu}_{40} - 3(\bar{\nu}_{20}^2 + \nu_{2:\nu_2}) = \bar{\lambda}_{40} - 3\nu_{2:\nu_2},$$

if $\nu_{2:\nu_2}$ is the variance of ν_2 in the grouped distributions.

In a similar way one can obtain such formulae for seminvariants as may be required. Through the sixth, the formulae for the Sheppard's corrections for the seminvariants calculated from a grouped distribution of a discrete variate are:

$$\lambda_2 = l_2 + \bar{\lambda}_{02} - \left(1 - \frac{1}{m^2}\right)\frac{k^2}{12}$$

$$\lambda_3 = l_3 + 3\bar{\lambda}_{21} + \bar{\lambda}_{03}$$

$$\lambda_4 = l_4 + 3\nu_{2:\nu_2} + 4\bar{\lambda}_{31} + 6\bar{\lambda}_{22} + \bar{\lambda}_{04} + \left(1 - \frac{1}{m^4}\right)\frac{k_4}{120} \tag{13}$$

$$\lambda_5 = l_5 + 10\nu_{11:\nu_2,\nu_3} + 5\bar{\lambda}_{41} + 10\bar{\lambda}_{32} + 10\bar{\lambda}_{23} + \bar{\lambda}_{05}$$

$$\lambda_6 = l_6 + 15\nu_{11:\nu_2,\nu_4} + 10\nu_{2:\nu_2} - 30\nu_{3:\nu_2} - 90\nu_{2:\nu_2}\bar{\nu}_{20}$$
$$+ 6\bar{\lambda}_{51} + 15\bar{\lambda}_{42} + 20\bar{\lambda}_{33} + 15\bar{\lambda}_{24} + \bar{\lambda}_{06} - \left(1 - \frac{1}{m^6}\right)\frac{k^6}{252}.$$

In these formulae, $\nu_{ij:\nu_r,\nu_s}$ is the ijth central product moment of ν_r and ν_s in the grouped distributions.

To illustrate these formulae numerically and to facilitate comparison with Pierce's results, we will use the example he chose. His ungrouped distribution was:

v	f	v	f	v	f
1	2	4	30	7	1
2	8	5	4	8	1
3	10	6	3	9	1

From this the following three grouped distributions with $k = 3$ can be formed:

(1)		(2)		(3)	
class	f	class	f	class	f
1– 3	20	0– 2	10	−1 [−1]	2
4–	37	3–	44	2–	48
7–	3	6–	5	5–	8
10–12	0	9–11	1	8–10	2

With origin at $\nu = 4$, we have the following table of moment characteristics of these four distributions:

Distribution	ν_1'	$\nu_2 = \bar{\lambda}_2$	$\nu_3 = \bar{\lambda}_3$	ν^4	$\bar{\lambda}^4$	$\delta\nu_1' = \nu_1' - \left(-\frac{10}{60}\right)$
(1)	$\frac{9}{60}$	$\frac{9819}{60^2}$	$-\frac{17442}{60^3}$	$\frac{238849317}{60^4}$	$-\frac{50388966}{60^4}$	$\frac{19}{60}$
(2)	$-\frac{9}{60}$	$\frac{10179}{60^2}$	$\frac{567162}{60^3}$	$\frac{557840277}{60^4}$	$\frac{247004154}{60^4}$	$\frac{1}{60}$
(3)	$-\frac{30}{60}$	$\frac{8820}{60^2}$	$\frac{1317600}{60^3}$	$\frac{528282000}{60^4}$	$\frac{294904800}{60^4}$	$-\frac{20}{60}$
Average	$-\frac{10}{60}$	$\frac{9606}{60^2}$	$\frac{622440}{60^3}$	$\frac{441657198}{60^4}$	$\frac{163839996}{60^4}$	
	μ_1'	$\mu_2 = \lambda_2$	$\mu_3 = \lambda_3$	μ_4	λ_4	
Original Distribution	$-\frac{10}{60}$	$\frac{7460}{60^2}$	$\frac{642400}{60^3}$	$\frac{305034000}{60^4}$	$\frac{138079200}{60^4}$	

From the table,

$$\bar{\nu}_{20} = \bar{\lambda}_{20} = \frac{9606}{60^2}$$

$$\bar{\nu}_{30} = \bar{\lambda}_{30} = \frac{622440}{60^3}$$

$$\bar{\nu}_{40} = \bar{\lambda}_{40} + 3\bar{\lambda}_{20}^2 = \frac{441657198}{60^4}.$$

We further compute:

$$\bar{\nu}_{02} = \frac{\Sigma(\delta\nu_1')^2}{3} = \frac{254}{60^2} = \bar{\lambda}_{02} \qquad \bar{\nu}_{21} = \frac{\Sigma(\nu_2\delta\nu_1')}{3} = \frac{6780}{60^3} = \bar{\lambda}_{21}$$

$$\bar{\nu}_{03} = \frac{-380}{60^3} = \bar{\lambda}_{03} \qquad \bar{\nu}_{30} = -\frac{8705412}{60^4} = \bar{\lambda}_{31}$$

$$\bar{\nu}_{04} = \frac{96774}{60^4} \qquad \bar{\nu}_{22} = \frac{2360946}{60^4}$$

$$\bar{\lambda}_{22} = \bar{\nu}_{22} - \bar{\nu}_{20}\bar{\nu}_{02} = \frac{-72978}{60^4}$$

$$\bar{\lambda}_{04} = \bar{\nu}_{04} - 3\bar{\nu}_{02}^2 = \frac{-96774}{60^4}$$

$$\nu_{2:\nu_2} = \frac{\Sigma\nu_2^2}{3} - \bar{\nu}_{20}^2 = \frac{330948}{60^4}$$

$$l_4 = \bar{\lambda}_{40} - 3\nu_{2:\nu_2} = \bar{\nu}_{40} - 3\bar{\nu}_{20}^2 - 3\nu_{2:\nu_2} = \frac{163839996}{60^4}$$

$$\left(1 - \frac{1}{m^2}\right)\frac{k^2}{12} = \frac{2}{3}$$

$$\left(1 - \frac{1}{m^2}\right)\left(7 - \frac{3}{m^2}\right)\frac{k^4}{240} = 2.$$

With these values one may check the formulae (12) and (13) as far as weight four. For example:

$$\mu_2 = \frac{9606}{60^2} + \frac{254}{60^2} - \frac{2}{3} = \frac{7460}{60^2}$$

$$\lambda_4 = \frac{1}{60^4}(163839996 + 991494 - 34821648 - 437868 - 96774 + 8640000)$$

$$= \frac{138079200}{60^4}.$$

It may appear at first glance that since

$$\bar{\nu}_{rs} = E[\nu_r(\delta\nu_1)^s]$$

and could be expressed by means of the notation, $\nu_{1s:\nu_r,\nu_1}'$, the notation in (12) and (13) could be made more uniform. It could be but at the expense of greater complexity in these two sets of results. Moreover, it is convenient that $\bar{\lambda}_{rs}$ is expressible in terms of $\bar{\nu}_{kl}$'s in precisely the same way that product seminvariants are ordinarily expressible in terms of product moments.

Pierce's results differ from the above not only in their mode of derivation but also in the fact that they express $\bar{\nu}_{r0}$'s and l_r's in terms of the characteristics of the ungrouped distribution and moments and seminvariants of moments in the grouped distributions. Thus as they stand they are not formulae for Sheppard's corrections.

Finally it must be remarked that in comparison with the usual formulae for Sheppard's corrections, the formulae (10) and (13) introduce quantities the magnitudes of which are not known in general except that ordinarily they are quite small. It is hoped that results on this point will be forthcoming soon.

REFERENCES

[1] H. C. Carver, "The fundamental nature and proof of Sheppard's adjustments," *Annals of Math. Stat.*, Vol. 7 (1936), pp. 154–163.

[2] J. A. Pierce, "A study of a universe of n finite populations with application to moment-function adjustments for grouped data," *Annals of Math. Stat.*, Vol. 11 (1940), pp. 311–334.

[3] C. C. Craig, "Sheppard's corrections for a discrete variable," *Annals of Math. Stat.*, Vol. 7 (1936), pp. 55–61.

ON THE ANALYSIS OF VARIANCE IN CASE OF MULTIPLE CLASSIFICATIONS WITH UNEQUAL CLASS FREQUENCIES

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In a previous paper[2] the author considered the case of a single criterion of classification with unequal class frequencies and derived confidence limits for σ'^2/σ^2 where σ'^2 denotes the variance associated with the classification, and σ^2 denotes the residual variance. The scope of the present paper is to extend those results to the case of multiple classifications with unequal class frequencies.

For the sake of simplicity of notations we will derive the required confidence limits in the case of a two-way classification, the extension to multiple classifications being obvious.

Consider a two-way classification with p rows and q columns. Let y be the observed variable, and let n_{ij} be the number of observations in the ith row and jth column. Denote by $y_{ij}^{(k)}$ the kth observation on y in the ith row and jth column ($k = 1, \cdots, n_{ij}$). Let the total number of observations be N. We order the N observations and let y_α be the αth observation on y in that order. Consider the variables:

$$t, t_1, \cdots, t_p, v_1, \cdots, v_q,$$

and denote by t_α the αth observation on t, by $t_{i\alpha}$ the αth observation on t_i and by $v_{j\alpha}$ the αth observation on v_j. The values of t_α, $t_{i\alpha}$ and $v_{j\alpha}$ are defined as follows:

$$t_\alpha = 1 \ (\alpha = 1, \cdots, N),$$

$$t_{i\alpha} = 1 \text{ if } y_\alpha \text{ lies in the } i\text{th row},$$

$$t_{i\alpha} = 0 \text{ if } y_\alpha \text{ does not lie in the } i\text{th row},$$

$$v_{j\alpha} = 1 \text{ if } y_\alpha \text{ lies in the } j\text{th column},$$

$$v_{j\alpha} = 0 \text{ if } y_\alpha \text{ does not lie in the } j\text{th column}.$$

We make the assumptions

$$y_{ij}^{(k)} = x_{ij}^{(k)} + \epsilon_i + \eta_j,$$

where the variates $x_{ij}^{(k)}$, ϵ_i, η_j ($i = 1, \cdots, p; j = 1, \cdots, q; k = 1, \cdots, n_{ij}$) are independently and normally distributed, the variance of $x_{ij}^{(k)}$ is σ^2, the variance of ϵ_i is σ'^2, the variance of η_j is σ''^2, and the mean values of ϵ_i and η_j are zero.

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[1] Research under a grant-in-aid from the Carnegie Corporation of New York.

[2] "A note on the analysis of variance with unequal class frequencies," *Annals of Math. Stat.*, Vol. 11 (1940).

Let the sample regression of y on $t, t_1, \cdots, t_{p-1}, v_1, \cdots, v_{q-1}$ be

$$Y = at + b_1t_1 + \cdots + b_{p-1}t_{p-1} + d_1v_1 + \cdots + d_{q-1}v_{q-1}.$$

We want to derive confidence limits for

$$\sigma'^2/\sigma^2 = \lambda^2.$$

Let us introduce the notations:

$$\begin{aligned}
\sum_\alpha t_\alpha t_{i\alpha} &= a_{0i} && (i = 1, \cdots, p-1),\\
\sum t_\alpha v_{j\alpha} &= a_{0p-1+j} && (j = 1, \cdots, q-1),\\
\sum t_{i\alpha} t_{j\alpha} &= a_{ij} && (i, j = 1, \cdots, p-1),\\
\sum t_{i\alpha} v_{j\alpha} &= a_{ip-1+j} && (i = 1, \cdots, p-1; j = 1, \cdots, q-1),\\
\sum v_{i\alpha} v_{j\alpha} &= a_{p-1+i\ p-1+j} && (i, j = 1, \cdots, q-1),\\
\| c_{ij} \| &= \| a_{ij} \|^{-1} && (i, j = 0, 1, \cdots, p+q-2).
\end{aligned}$$

Let the regression of $x_{ij}^{(k)}$ on $t, t_1, \cdots, t_{p-1}, v_1, \cdots, v_{q-1}$ be

$$X = a^*t + b_1^*t_1 + \cdots + b_{p-1}^*t_{p-1} + d_1^*v_1 + \cdots + d_{q-1}^*v_{q-1}.$$

The regression of $\epsilon_i + \eta_j$ on the same independent variables is evidently equal to

$$\begin{aligned}
&\epsilon_1t_1 + \cdots + \epsilon_pt_p + \eta_1v_1 + \cdots + \eta_qv_q\\
&\qquad = (\eta_q + \epsilon_p)t + (\epsilon_1 - \epsilon_p)t_1 + \cdots + (\epsilon_{p-1} - \epsilon_p)t_{p-1}\\
&\qquad\quad + (\eta_1 - \eta_q)v_1 + \cdots + (\eta_{q-1} - \eta_q)v_{q-1},
\end{aligned}$$

since $t_p = t - t_1 - \cdots - t_{p-1}$ and $v_q = t - v_1 - \cdots - v_{q-1}$. Hence

$$b_i = b_i^* + (\epsilon_i - \epsilon_p), \qquad (i = 1, \cdots, p-1), \tag{1}$$

and therefore

$$\begin{aligned}
\sigma_{b_ib_j} = \sigma_{b_i^*b_j^*} + \sigma_{(\epsilon_i-\epsilon_p)(\epsilon_j-\epsilon_p)} &= c_{ij}\sigma^2 + \sigma_{\epsilon_i\epsilon_j} + \sigma_{\epsilon_p\epsilon_p}\\
&= [c_{ij} + (1 + \delta_{ij})\lambda^2]\sigma^2, \quad (i, j = 1, \cdots, p-1),
\end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta, i.e. $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Denote $c_{ij} + (1 + \delta_{ij})\lambda^2$ by c'_{ij}. Since the expected value of b_i^* is equal to zero, on account of (1) also the expected value of b_i is equal to zero. Let

$$\| g_{ij} \| = \| c'_{ij} \|^{-1}, \qquad (i, j = 1, \cdots, p-1).$$

Then

$$\frac{1}{\sigma^2} \sum_{j=1}^{p-1} \sum_{i=1}^{p-1} g_{ij} b_i b_j \tag{3}$$

has the χ^2-distribution with $p - 1$ degrees of freedom. The expression

$$\frac{1}{\sigma^2} \sum_{\alpha=1}^{N} (y_\alpha - Y_\alpha)^2, \tag{4}$$

has the χ^2-distribution $N - p - q + 1$ degrees of freedom. The expressions (3) and (4) are independently distributed. Hence

$$\frac{N - p - q + 1}{p - 1} \frac{\Sigma\Sigma g_{ij} b_i b_j}{\Sigma (y_\alpha - Y_\alpha)^2}, \tag{5}$$

has the F-distribution (analysis of variance distribution). We will now show that (5) is a monotonic function of λ^2. It is known that $\Sigma\Sigma g_{ij} b_i b_j$ is invariant under linear transformations, i.e.

$$\Sigma\Sigma g_{ij} b_i b_j = \Sigma\Sigma g'_{ij} b'_i b'_j ,$$

where b'_i is an arbitrary linear function, say $\mu_{i1} b_1 + \cdots + \mu_{ip-1} b_{p-1}$ of $b_1, \cdots, b_{p-1}$ $(i = 1, \cdots, p - 1)$ and

$$|| g'_{ij} || = || \sigma_{b'_i b'_j} ||^{-1}.$$

We can choose the matrix $|| \mu_{ij} ||$ such that

$$\epsilon'_i = \mu_{i1}(\epsilon_1 - \epsilon_p) + \cdots + \mu_{ip-1}(\epsilon_{p-1} - \epsilon_p), \qquad (i = 1, \cdots, p - 1),$$

are independently distributed and $\sigma^2_{\epsilon'_i} = \sigma'^2$. The coefficients μ_{ij} of course do not depend on σ'. We have

$$\sigma_{b'_i b'_j} = \sigma_{b^*_i b^*_j} + \delta_{ij} \sigma'^2, \qquad (\delta_{ij} = \text{Kronecker delta}).$$

Now let

$$b''_i = \nu_{i1} b'_1 + \cdots + \nu_{ip-1} b'_{p-1}, \qquad (\nu = 1, \cdots, p - 1),$$

where $|| \nu_{ij} ||$ is an orthogonal matrix and is chosen such that $b^{*}_1, \cdots, b^{*}_{p-1}$ are independently distributed. On account of the orthogonality of $|| \nu_{ij} ||$ we obviously have

$$\sigma^2_{b''_i} = \sigma^2_{b^{*}_i} + \sigma'^2 ; \qquad \sigma_{b''_i b''_j} = 0 \qquad \text{for } i \neq j.$$

Hence

$$\sum \sum g_{ij} b_i b_j = \sum_{i=1}^{p-1} \frac{b''^2_i}{\sigma^2_{b^{*}_i} + \lambda^2 \sigma^2}. \tag{6}$$

The right hand side of (6) is evidently a monotonic function of λ^2 which proves our statement. The endpoints of the confidence interval for λ^2 are the roots in λ^2 of the equations

$$\frac{N - p - q + 1}{p - 1} \frac{\Sigma\Sigma g_{ij} b_i b_j}{\Sigma (y_\alpha - Y_\alpha)^2} = F_2 ; \qquad \frac{N - p - q + 1}{p - 1} \frac{\Sigma\Sigma g_{ij} b_i b_j}{\Sigma (y_\alpha - Y_\alpha)^2} = F_1, \tag{7}$$

where F_2 denotes the upper, and F_1 the lower critical value of F.

The derivation of the required confidence limits in case of classifications in more than two ways can be carried out in the same way and I shall merely state here the results.

Consider r criterions of classifications and denote by p_u the number of classes in the uth classification ($u = 1, \cdots, r$). Denote by $n_{i_1 \cdots i_r}$ the number of observations which belong to the i_1th class of the first classification, i_2th class of the second classification, $\cdots$, and to the i_rth class of the rth classification. Let $y^{(k)}_{i_1 \cdots i_r}$ be the kth observation on y in the set of observations belonging to the classes mentioned above ($k = 1, \cdots, n_{i_1 \cdots i_r}$). We make the assumption

$$y^{(k)}_{i_1 \cdots i_r} = x^{(k)}_{i_1 \cdots i_r} + \epsilon^{(1)}_{i_1} + \cdots + \epsilon^{(r)}_{i_r},$$

where the variates

$$x^{(k)}_{i_1 \cdots i_r}, \ \epsilon^{(1)}_{i_1}, \cdots, \epsilon^{(r)}_{i_r} \quad (i_u = 1, \cdots, p_u ; \ u = 1, \cdots, r; \ k = 1, \cdots, n_{i_1 \cdots i_r}),$$

are independently and normally distributed, the variance of $x^{(k)}_{i_1 \cdots i_r}$ is σ^2, the variance of $\epsilon^{(u)}_{i_u}$ is σ^2_u and the mean value of $\epsilon^{(u)}_{i_u}$ is zero ($i_u = 1, \cdots, p_u$; $u = 1, \cdots, r$).

Let N be the total number of observations. We order the observations in a certain order and denote by y_α the αth observation in that order ($\alpha = 1, \cdots, N$). Consider the variables:

$$t, \ t^{(u)}_{i_u}, \qquad (u = 1, \cdots, r; \ i_u = 1, \cdots, p_u),$$

and denote by t_α the αth observation on t and by $t^{(u)}_{i_u}\alpha$ the αth observation on $t^{(u)}_{i_u}$. The values of t_α and $t^{(u)}_{i_u \alpha}$ are given as follows:

$t_\alpha = 1$ ($\alpha = 1, \cdots, N$),

$t^{(u)}_{i_u \alpha} = 1$ if y_α lies in the i_uth class of the uth classification,

$t^{(u)}_{i_u \alpha} = 0$ if y_α does not lie in the i_uth class of the uth classification.

Let the sample regression of y on t, $t^{(u)}_{i_u}$ be given by

$$Y = at + \sum_{u=1}^{r} \sum_{i_u=1}^{p_u - 1} b^{(u)}_{i_u} t^{(u)}_{i_u}.$$

Let the covariance of $b^{(u)}_{i_u}$ and $b^{(u)}_{j_u}$ be given by $C^{(u)}_{i_u j_u} \sigma^2$ under the assumption that $\sigma_1 = \sigma_2 = \cdots = \sigma_r = 0$. The matrix $|| \, C^{(u)}_{i_u j_u} \, ||$ ($i_u, j_u = 1, \cdots, p_u - 1$) can be calculated by known methods of the theory of least squares. Let

$$|| \, g^{(u)}_{i_u j_u} \, || = || \, C^{(u)}_{i_u j_u} + (1 + \delta_{i_u j_u}) \lambda^2_u \, ||^{-1} \qquad (i_u, j_u = 1, \cdots, p_u - 1),$$

where $\delta_{i_u j_u}$ is the Kronecker delta and $\lambda^2_u = \sigma^2_u / \sigma^2$. Then the lower and upper confidence limits for λ^2_u are given by the roots in λ^2_u of the equations

$$\frac{N - \sum_{u=1}^{r} p_u + r - 1}{p_u - 1} \cdot \frac{\sum_{j_u=1}^{p_u - 1} \sum_{i_u=1}^{p_u - 1} g^{(u)}_{i_u j_u} b^{(u)}_{i_u} b^{(u)}_{j_u}}{\sum_{\alpha=1}^{N} (y_\alpha - Y_\alpha)^2} = F_i \qquad (i = 1, 2), \tag{8}$$

where F_2 is the upper and F_1 the lower critical value of the analysis of variance distribution with $p_u - 1$ and $N - \sum_{u=1}^{r} p_u + r - 1$ degrees of freedom. In case of a single criterion of classification the confidence limits (8) are identical with those given in my previous paper.

THE FREQUENCY DISTRIBUTION OF A GENERAL MATCHING PROBLEM

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1. Introduction. This paper considers the matching of two decks of cards of arbitrary composition, and the complete frequency distribution of correct matchings is obtained, thus solving a problem proposed by Stevens.[1] It is also shown that the results can be interpreted in terms of a contingency table.

Generalizing a problem considered by Greenwood,[2] let us consider the matching of two decks of cards consisting of t distinct kinds, all the cards of each kind being identical. The first or "call" deck will be composed of i_1 cards of the first kind, i_2 of the second, etc., such that

$$i_1 + i_2 + i_3 + \cdots + i_t = n;$$

and the second or "target" deck will contain j_1 cards of the first kind, j_2 of the second, etc., such that

$$j_1 + j_2 + \cdots + j_t = n.$$

Any of the i's or j's may be zero. It is desired to calculate, for a given arrangement of the "call" deck, the number of possible arrangements of the "target" deck which will produce exactly r matchings between them ($r = 0, 1, 2, \cdots, n$). It is clear that these frequencies are independent of the arrangement of the call deck. For convenience the call deck may be thought of as arranged so that all the cards of the first kind come first, followed by all those of the second kind, and so on.

2. Formulae for the frequencies. Let us consider the number of arrangements of the target deck which will match the cards in the k_1th, k_2th, $\cdots$, k_sth positions in the call deck, regardless of whether or not matchings occur elsewhere. Let the cards in these s positions in the call deck consist of c_1 of the first kind, c_2 of the second, etc. Then:

$$c_1 + c_2 + \cdots + c_t = s.$$

The number of such arrangements of the target deck is

$$(1) \qquad \frac{(n-s)!}{\prod_{h=1}^{t} (j_h - c_h)!}.$$

[1] W. L. Stevens, *Annals of Eugenics*, Vol. 8 (1937), pp. 238–244.

[2] J. A. Greenwood, *Annals of Math. Stat.*, Vol. 9 (1938), pp. 56–59.

For fixed values of the c's, the s specified positions may be selected in

$$\prod_{h=1}^{t} \frac{i_h!}{c_h!(i_h - c_h)!} \tag{2}$$

ways.

Consider now the expression

$$V_s = \sum \frac{(n-s)! \prod_{h=1}^{t} i_h!}{\prod_{h=1}^{t} c_h!(i_h - c_h)!(j_h - c_h)!} \tag{3}$$

obtained by summing the product of (1) and (2) over all sets of values of the numbers $c_1, c_2 \cdots, c_t$ satisfying the conditions:

$$0 \leqq c_h \leqq i_h, \qquad c_h \leqq j_h, \qquad \text{and} \qquad \sum_{h=1}^{t} c_h = s.$$

Let W_s denote the number of arrangements of the target deck which result in exactly s matchings. Then it is evident that V_s exceeds W_s, since the former includes those arrangements which give more than s matchings, and these, moreover, are counted more than once. Consider an arrangement which produces u matchings, where $u > s$. Such an arrangement will be counted once in V_s for every set of s matchings which can be selected from the total of u—that is ${}^uC_s$ times. In other words,

$$V_r = W_r + {}^{r+1}C_r W_{r+1} + {}^{r+2}C_r W_{r+2} + \cdots + {}^nC_r W_n.$$

It has been shown[3] that the solution of these equations is

$$W_r = V_r - {}^{r+1}C_r V_{r+1} + {}^{r+2}C_r V_{r+2} - \cdots + (-1)^{n-r}\, {}^nC_r V_n. \tag{4}$$

3. Computation of the frequencies. Equations (3) and (4) apparently give the solution of the problem, but in practice the labor of carrying out the summation indicated in (3) would often be very great. However, (3) may be rewritten in the form

$$V_s = \frac{(n-s)!}{\prod_{h=1}^{t} j_h!} H_s, \tag{5}$$

where

$$H_s = \sum \left\{ \prod_{h=1}^{t} \frac{i_h!\, j_h!}{c_h!\,(i_h - c_h)!\,(j_h - c_h)!} \right\}.$$

[3] H. GEIRINGER, *Annals of Math. Stat.*, Vol. 9 (1938), p. 262.

It will be seen that H_s is the coefficient of x^s in the product

$$\prod_{h=1}^{t}\left\{\sum_{k=0}^{i_h'}\frac{i_h!\,j_h!\,x^k}{k!\,(i_h-k)!\,(j_h-k)!}\right\}, \tag{6}$$

where i_h' denotes the smaller of i_h and j_h. The factor $\prod_{h=1}^{t} j_h!$ was included in H_s in order to make the coefficients in the polynomials of (6) always integers.

Equation (4) may now be written in the form

$$W_r = \sum_{s=r}^{n}(-1)^{s-r}\,{}^sC_r\,\frac{(n-s)!}{\prod_{h=1}^{t} j_h!}\,H_s,$$

or

$$W_r = \frac{1}{r!}\sum_{s=r}^{n}\frac{(-1)^{s-r}}{(s-r)!}\,\frac{s!\,(n-s)!}{\prod_{h=1}^{t} j_h!}\,H_s, \tag{7}$$

a form which lends itself to actual computation.

4. Factorial moments. The factorial moments of the frequency distribution of the number of matchings are easy to compute. Let m_s denote the sth factorial moment, so that

$$m_s = \frac{\sum_{r=s}^{n} r^{(s)}\,W_r}{\sum_{r=0}^{n} W_r}. \tag{8}$$

Substituting from (4)

$$\sum_{r=s}^{n} r^{(s)}W_r = \sum_{r=s}^{n}\left\{r^{(s)}\sum_{u=r}^{n}(-1)^{u-r}\,{}^uC_r\,V_u\right\}.$$

Reversing the order of summation and simplifying,

$$\sum_{r=s}^{n} r^{(s)}W_r = \sum_{u=s}^{n}\left\{u^{(s)}\,V_u\sum_{r=s}^{u}(-1)^{u-r}\,{}^{u-s}C_{r-s}\right\} = s!\,V_s.$$

Hence,

$$V_0 = \sum_{r=0}^{n} W_r = \frac{n!}{\prod_{h=1}^{t} j_h!}, \tag{9}$$

and from (5) and (8),

$$m_s = \frac{H_s}{{}_nC_s}. \tag{10}$$

5. Mean and variance. From (6)

$$H_1 = \sum_{h=1}^{t} i_h j_h \tag{11}$$

and

$$H_2 = \tfrac{1}{2} \sum_{h=1}^{t} i_h(i_h - 1) j_h(j_h - 1) + \sum_{\substack{h,k=1 \\ h \neq k}}^{t} i_h i_k j_h j_k. \tag{12}$$

Hence the mean number of matchings is

$$m_1 = \frac{\sum_{h=1}^{t} i_h j_h}{n}. \tag{13}$$

The variance μ_2 is

$$m_2 + m_1 - m_1^2 = \frac{1}{n^2(n-1)} \left[n \sum_{h=1}^{t} i_h(i_h - 1) j_h(j_h - 1) + 2n \sum_{\substack{h,k=1 \\ h<k}}^{t} i_h i_k j_h j_k \right.$$
$$\left. + n(n-1) \sum_{h=1}^{t} i_h j_h - (n-1) \left(\sum_{h=1}^{t} i_h j_h \right)^2 \right],$$

or

$$\mu_2 = \frac{1}{n^2(n-1)} \left\{ \left(\sum_{h=1}^{t} i_h j_h \right)^2 - n \sum_{h=1}^{t} i_h j_h (i_h + j_h) + n^2 \sum_{h=1}^{t} i_h j_h \right\}. \tag{14}$$

In the special case $j_1 = j_2 = \cdots = j_t = j$, these formulae become

$$M_1 = j, \qquad \mu_2 = \frac{j}{n(n-1)} \left(n^2 - \sum_{h=1}^{t} i_h^2 \right).$$

These formulae have previously been given by Stevens,[4] and those for the special case also by Greenwood. The maximal conditions for the variance, given by Greenwood for this particular case, apparently can not be put in a simple form for the general case.

6. Unequal decks. Suppose the call deck contains m cards, $m < n$, and is to be matched with m cards selected from the target deck. It can be assumed without loss of generality that the first m cards in any arrangement of the target deck are the ones to be used. The formulae of this paper can be applied to this

[4] W. L. Stevens, *Annals of Eugenics*, loc. cit., *Psychol. Review*, Vol. 46 (1939), pp. 142–150.

more general problem by the expedient of imagining $n - m$ blank cards to be added at the end of the call deck and regarding these as an additional kind. It is thus apparent that formulae (13) and (14) apply without modification to this altered situation.

7. Application to contingency table. Stevens[5] has considered the distribution of entries in a contingency table with fixed marginal totals, and has pointed out that the problem of matching two decks of cards may be dealt with from that standpoint. A contingency table classifies data into n columns and m rows, and we may consider the row as indicating the kind of card which occupies a given position in the call deck, the columns having the same function with respect to the target deck. Stevens defines a quantity c as the sum of entries in a prescribed set of cells, subject to the condition that no two cells of the set are in the same row or column, and mentions as unsolved the problem of the exact sampling distribution of c.

We now have at our disposal the machinery for solving this problem. Following Stevens's notation, let $a_1, a_2, \cdots, a_m$ denote the fixed row totals and $b_1, b_2, \cdots, b_n$ the fixed column totals, while x_{rs} denotes the frequency of the cell in the rth row and the sth column. Then, let $c = \sum_{h=1}^{l} x_{r_h s_h}$, where l does not exceed either m or n. Imagine two decks of N cards $\left(N = \sum_{h=1}^{m} a_h = \sum_{h=1}^{n} b_h\right)$, the first containing a_1 cards of one kind, a_2 of another, etc., and the second containing b_1 cards of one kind, b_2 of another, etc. Moreover, let the r_hth kind in the first deck and the s_hth kind in the second deck be the same kind ($h = 1, 2, \cdots, l$), the other kinds being all different. Evidently c is the number of matchings between the two decks. Hence, the methods of this paper can be used to obtain the distribution of c. The formulae we have obtained agree with those for the expected value and variance of c given by Stevens.

ON METHODS OF SOLVING NORMAL EQUATIONS

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There seems to be considerable disagreement concerning what is the most satisfactory method of solving a set of normal equations. Since such information as errors of estimate and significance of results is usually desired in addition to the solution, in its broader aspects the problem is one of deciding what is the most satisfactory method of calculating the inverse of a symmetric matrix.

For equations with several unknowns some compact systematic method of

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[5] W. L. Stevens, *Annals of Eugenics*, loc. cit.

calculation is necessary to eliminate much of the labor involved in the ordinary method of calculating the inverse from its definition. Among the more common of such systematic methods are those associated with the names of Chio,[1] Gauss,[1] Doolittle,[2] and Aitken.[3] In addition, A. A. Albert[4] recently called attention to a method implicit in elementary matrix theory. There are also various iterative schemes, and schemes which are but slight variations of the above methods. In this note only the methods associated with the above names will be considered, and for convenience they will be labeled with those names, regardless of who should be given credit for them.

The purpose of this note is to show that when the calculation of the inverse is systematized, all of the above methods are fundamentally equivalent and merely involve a different arrangement of work. Consequently, any advantage in calculating time for any particular method will arise through such features as a simpler technique or less copying, rather than through fewer multiplications and divisions.

By the method of Chio is meant the evaluation of determinants by the pivotal method of reduction. Since all of the methods mentioned above use pivotal reduction, the method of Chio will not be treated as a distinct method. Furthermore, since Gauss' method is incorporated in that of Aitken, it will be necessary to consider only the methods of Aitken, Doolittle, and Albert as distinct.

First consider the method of Albert, which is based on the following matrix properties. Let the matrix $\mathbf{A}$ be subjected to a sequence of row transformations leading to the matrix $\mathbf{A}'$. Then, writing $\mathbf{A} = \mathbf{IA}$, it follows from a theorem in matrix theory that $\mathbf{A}' = \mathbf{I}'\mathbf{A}$, and consequently that $\mathbf{A}'\mathbf{A}^{-1} = \mathbf{I}'$. If row transformations are chosen which make $\mathbf{A}' = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{I}'$. This states that if the same row transformations are applied to the identity matrix as were used to reduce $\mathbf{A}$ to the identity matrix, then the resulting matrix will be the desired inverse. The customary manner of reducing $\mathbf{A}$ to $\mathbf{I}$ is to work for zeros in columns as follows:

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix}, \qquad a_{11}\begin{Vmatrix} 1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ 0 & \left(a_{22} - a_{12}\frac{a_{21}}{a_{11}}\right) & \cdots & \left(a_{2n} - a_{1n}\frac{a_{21}}{a_{11}}\right) \\ \vdots & \vdots & & \vdots \\ 0 & \left(a_{n2} - a_{12}\frac{a_{n1}}{a_{11}}\right) & \cdots & \left(a_{nn} - a_{1n}\frac{a_{n1}}{a_{11}}\right) \end{Vmatrix},$$

[1] See, for example, Whittaker and Robinson, *The Calculus of Observations*, p. 71 and p. 234.

[2] See, for example, Croxton and Cowden, *Applied General Statistics*, 1939, p. 716.

[3] *Roy. Soc. Edin. Proc.*, Vol. 57 (1936-37), p. 172.

[4] Am. Math. Monthly, Vol. 48, No. 3 (1941), p. 198.

$$a_{11}b_{22}\left\|\begin{matrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ 0 & 1 & \frac{b_{23}}{b_{22}} & \cdots & \frac{b_{2n}}{b_{22}} \\ 0 & 0 & \left(b_{33}-b_{23}\frac{b_{32}}{b_{22}}\right) & \cdots & \left(b_{3n}-b_{2n}\frac{b_{32}}{b_{22}}\right) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \left(b_{n3}-b_{23}\frac{b_{n2}}{b_{22}}\right) & \cdots & \left(b_{nn}-b_{2n}\frac{b_{n2}}{b_{22}}\right) \end{matrix}\right\|, \cdots,$$

where new letters are introduced for new elements after each reduction. After zeros are obtained below the main diagonal, zeros are obtained above the diagonal by starting with the last column. If now these operations are performed in the same order on $\mathbf{I}$, the result will be $\mathbf{A}^{-1}$.

Next consider the method of Aitken, which is based on the evaluation of a bordered determinant, namely,

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} & 1 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} & 0 \\ 0 & \cdots & -1 & \cdots & 0 & 0 \end{vmatrix} = \text{cofactor of } a_{ij}.$$

To obtain $\mathbf{A}^{-1}$ it is merely necessary to evaluate determinants of this type and divide them by $|A|$. Aitken's method evaluates all such determinants simultaneously, using Chio's reduction technique in much the same manner as illustrated above with Albert's method. Thus,

$$\left\|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \\ \hline -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{matrix}\right\|,$$

$$a_{11}\left\|\begin{array}{cccc|ccc}
1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} & \frac{1}{a_{11}} & 0 \cdots & 0 \\
0 & \left(a_{22}-a_{12}\frac{a_{21}}{a_{11}}\right) & \cdots & \left(a_{2n}-a_{1n}\frac{a_{21}}{a_{11}}\right) & -\frac{a_{21}}{a_{11}} & 1 \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & \left(a_{n2}-a_{12}\frac{a_{n1}}{a_{11}}\right) & \cdots & \left(a_{nn}-a_{1n}\frac{a_{n1}}{a_{11}}\right) & -\frac{a_{n1}}{a_{11}} & 0 \cdots & 1 \\
\hline
0 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} & \frac{1}{a_{11}} & 0 \cdots & 0 \\
0 & -1 & \cdots & 0 & 0 & 0 \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & -1 & 0 & 0 \cdots & 0
\end{array}\right\|,$$

$$a_{11}b_{22}\left\|\begin{array}{cccc|ccc}
1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots \frac{a_{1n}}{a_{11}} & \frac{1}{a_{11}} & 0 & \cdots 0 \\
0 & 1 & \frac{b_{23}}{b_{22}} & \cdots \frac{b_{2n}}{b_{22}} & -\frac{a_{21}}{a_{11}b_{22}} & \frac{1}{b_{22}} & \cdots 0 \\
0 & 0 & \left(b_{33}-b_{23}\frac{b_{32}}{b_{22}}\right) & \cdots \left(b_{3n}-b_{2n}\frac{b_{32}}{b_{22}}\right) & \left(\frac{a_{21}}{a_{11}}\frac{b_{32}}{b_{22}}-\frac{a_{31}}{a_{11}}\right) & -\frac{b_{32}}{b_{22}} & \cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \left(b_{n3}-b_{23}\frac{b_{n2}}{b_{22}}\right) & \cdots \left(b_{nn}-b_{2n}\frac{b_{n2}}{b_{22}}\right) & \left(\frac{a_{21}}{a_{11}}\frac{b_{n2}}{b_{22}}-\frac{a_{n1}}{a_{11}}\right) & -\frac{b_{2n}}{b_{22}} & \cdots 1 \\
\hline
0 & 0 & \left(\frac{a_{13}}{a_{11}}-\frac{a_{12}}{a_{11}}\frac{b_{23}}{b_{22}}\right) & \cdots \left(\frac{a_{1n}}{a_{11}}-\frac{a_{12}}{a_{11}}\frac{b_{2n}}{b_{22}}\right) & \left(\frac{a_{21}^2}{a_{11}^2b_{22}}+\frac{1}{a_{11}}\right) & -\frac{a_{12}}{a_{11}b_{22}} & \cdots 0 \\
0 & 0 & \frac{b_{23}}{b_{22}} & \cdots \frac{b_{2n}}{b_{22}} & -\frac{a_{12}}{a_{11}b_{22}} & \frac{1}{b_{22}} & \cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots -1 & 0 & 0 & \cdots 0
\end{array}\right\|, \cdots.$$

When zeros are obtained below the main diagonal to the left of the vertical dividing line, the matrix in the lower right section will be $\mathbf{A}^{-1}$. This follows from the fact that the elements of this matrix will be the evaluations of bordered determinants, like those of the previous paragraph, divided by $a_{11}b_{22}\cdots = |A|$.

It will be observed that the operations on **A** in Albert's method which produce zeros below the main diagonal are the same as those which occur above the horizontal dividing line in Aitken's method. This set of operations is performed simultaneously on **I**, since the upper right section of Aitken's scheme is **I**. Furthermore, obtaining a zero for an element below the horizontal line and to the left of the vertical line, is equivalent to obtaining a zero for the element corre-

sponding to the same row and column in the section above the horizontal, provided the preceding columns contain zeros above the diagonal. But obtaining zeros above the main diagonal of **A** constitutes the second set of operations in Albert's method to obtain $\mathbf{A}' = \mathbf{I}$. Thus, the operations in Aitken's method which produce zeros in a given column for elements above the horizontal line are merely the first set of operations in Albert's method, while those which produce zeros below the horizontal line are the second set of operations in reverse order. Since, in Aitken's scheme, the first set of operations is performed on **I** in the upper right section and the results are transferred a row at a time to the lower right section, where they are in turn operated upon by the second set of operations, this lower right section is merely **I** operated upon by the entire set of operations of Albert's method. Consequently, Aitken's and Albert's methods are the same except for the order in which operations are performed and differences arising therefrom. Since Aitken's method performs these operations more compactly, it is to be preferred to that of Albert.

Next consider the method of Doolittle, which is described by following the instructions given in the first column in the table shown on page 348. The forward solution is completed after n such sectional operations. For a given k column, the backward solution is obtained as usual by substitution in the last row of each section taken in reverse order.

If all summations in each section are performed in pairs and the sums recorded each time, rather than being performed in one operation, the forward solution of the Doolittle method will be found to be a rearrangement of the work occurring above the horizontal line in Aitken's method. Thus the first lines of each section give the matrix above the horizontal line in Aitken's scheme. Then, except for signs, I' and the sums of the first two lines of the remaining sections give the result of Aitken's first sequence of operations above the horizontal. Then, except for signs, II' and the sums of the first three lines of the remaining sections give the result of Aitken's second sequence of operations above the horizontal, etc.

The back solution involves precisely the same operations as those making up the second set of Albert's sequence of operations to obtain zeros above the main diagonal. Since these were shown to be a rearrangement of operations in Aitken's method, it follows that the methods of Aitken and Doolittle are the same except for the order of operations and differences arising therefrom. Hence all three methods are basically the same when systematized for a calculating machine.

Because of this equivalence, the number of necessary multiplications and divisions will be the same for all three methods, and will be found to be $\frac{1}{2}n^2(n + 1)$. Since Aitken's method is to be preferred to that of Albert, it will suffice to compare the methods of Aitken and Doolittle for calculating convenience.

The Doolittle method possesses several distinct advantages. First, its multiplications occur a row at a time with one of the factors constant for that row; consequently the keyboard remains unchanged for a given row of operations.

	1	2	3	$\cdots$	n	k_1	k_2	$\cdots$	k_n
I and ΣI	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	-1	0		0
I′	-1	$-\frac{a_{12}}{a_{11}}$	$-\frac{a_{13}}{a_{11}}$	$\cdots$	$-\frac{a_{1n}}{a_{11}}$	$\frac{1}{a_{11}}$	0	$\cdots$	0
II	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	0	-1		0
ΣI·I′$_2$	$-a_{12}$	$-a_{12}\frac{a_{12}}{a_{11}}$	$-a_{13}\frac{a_{12}}{a_{11}}$	$\cdots$	$-a_{1n}\frac{a_{12}}{a_{11}}$	$\frac{a_{12}}{a_{11}}$	0		0
ΣII	0	$\left(a_{22}-a_{12}\frac{a_{12}}{a_{11}}\right)$	$\left(a_{23}-a_{13}\frac{a_{12}}{a_{11}}\right)$	$\cdots$	$\left(a_{2n}-a_{1n}\frac{a_{12}}{a_{11}}\right)$	$\frac{a_{12}}{a_{11}}$	-1	$\cdots$	0
II′	0	-1	$-\frac{b_{23}}{b_{22}}$	$\cdots$	$-\frac{b_{2n}}{b_{22}}$	$-\frac{a_{12}}{a_{11}b_{22}}$	$\frac{1}{b_{22}}$		0
III	a_{31}	a_{32}	a_{33}	$\cdots$	a_{3n}	0	0		0
ΣI·I′$_3$	$-a_{13}$	$-a_{12}\frac{a_{13}}{a_{11}}$	$-a_{13}\frac{a_{13}}{a_{11}}$	$\cdots$	$-a_{1n}\frac{a_{13}}{a_{11}}$	$\frac{a_{13}}{a_{11}}$	0		0
ΣII·II′$_3$	0	$-b_{23}$	$-b_{23}\frac{b_{23}}{b_{22}}$	$\cdots$	$-b_{2n}\frac{b_{23}}{b_{22}}$	$-\frac{a_{12}}{a_{11}}\frac{b_{23}}{b_{22}}$	$\frac{b_{23}}{b_{22}}$	$\cdots$	0
ΣIII	0	0	c_{33}	$\cdots$	c_{3n}	c_{3n+1}	$\frac{b_{23}}{b_{22}}$		0
III′	0	0	-1	$\cdots$	$-\frac{c_{3n}}{c_{33}}$	$-\frac{c_{3n+1}}{c_{33}}$	$-\frac{b_{23}}{b_{22}c_{33}}$		0
$\vdots$			$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$

Aitken's method, however, consists of calculating successive cross products' which requires clearing of the keyboard after each such operation. Secondly, there are fewer additions in the Doolittle method. It sums i quantities at a time in section i, while Aitken's cross products always involve the sum of two quantities. Because of the necessity of calculating the complements of negative sums, this difference becomes important when the number of variables is large. A third feature in favor of the Doolittle method is the ease of performing the calculations without previous experience. It may be easier to understand how to calculate cross products, but actually the calculations of the Doolittle method are easier to perform. Aitken's method requires some experience with it, if one is to avoid repeating certain calculations which would result from calculating all cross products mechanically. The comparative amount of copying in the two methods depends upon the number of variables involved.

From the above considerations, it may be concluded that the Doolittle method is to be preferred among those considered in this paper for solving a set of normal equations or calculating the inverse of a symmetric matrix. However, if a single calculating technique is desired which can be used for nonsymmetrical equations as well, then the method of Aitken is to be preferred.

CONDITIONS THAT THE ROOTS OF A POLYNOMIAL BE LESS THAN UNITY IN ABSOLUTE VALUE

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1. Introduction. In econometric business cycle analysis, probability theory, and numerical mathematical computation the problem of convergence of repeated iterations arises. The solution of the difference equations defining such a process can in a wide variety of cases be shown to be stable in the sense of converging to a limit if a certain associated polynomial

$$f(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n = 0, \tag{1}$$

has roots whose moduli are all less than unity.

Thus, for "timeless" linear difference equation systems of the most general type, convertible into normal form,

$$Q_i(t+1) = \sum_{j=1}^{n} a_{ij}Q_j(t), \qquad (i = 1, \cdots, n), \tag{2}$$

the polynomial is the characteristic or determinantal equation,

$$f(x) = | a_{ij} - x\delta_{ij} | = 0, \tag{3}$$

which when expanded out is of the form (1). The roots of this equation, when multiplied by suitable polynomials in t, give the exact solution of the problem in the form

$$Q(t) = \sum_{i=1}^{m} g_i(t)x_i^t, \tag{4}$$

where m is the number of distinct roots, and the g's are polynomials of degree one less than the multiplicity of the respective root. If complex roots occur, they do so in conjugate pairs and can be combined to form damped, undamped, or anti-damped harmonic terms. All terms go to zero as t approaches infinity if, and only if, the absolute value of each x is less than unity.

For non-linear systems the exact solution does not take this form, but in the neighborhood of an equilibrium point the roots of an associated polynomial, except in singular cases, do determine the stability of the system.

As far as the writer is aware, there does not appear in the literature an account of necessary and sufficient conditions for the roots of a polynomial to be less than unity in absolute value. This is in contrast to a related problem which arises in connection with the investigation of stability of dynamical systems defined by differential equations. These have associated with them a polynomial whose roots provide solutions in the form

$$g_i(t)e^{x_it}, \tag{5}$$

or for non-linear systems infinite power series in such terms. It is required, therefore, to determine complete conditions under which the *real parts* of all roots must be negative.

This problem has been solved by Routh[1] in a manner which leaves little to be desired. Determinantal expression of his conditions in a slightly modified form was made by Hurwitz[2] who apparently was unaware of Routh's work, and by Frazer and Duncan[3] who were unaware of the Hurwitz results. A brief outline of Routh's mode of attack will prove instructive in dealing with the problem at hand.

2. Routhian analysis of sign of real parts of roots. Routh realized that the condition that all coefficients be positive—the leading coefficient having been made so—was necessary, but not sufficient unless all the roots were real. But a "derived" equation of degree $n(n-1)/2$ whose roots equal the sums of the roots of the original equation taken two at a time has real roots which are simple sums of the real parts of those of the original equation. In consequence, it is necessary and sufficient that the coefficients of the original and the "derived" equation all be positive.

Thus, valid necessary and sufficient conditions are presented. However, they are disadvantageous from two points of view. First, they are not all independent, being $n(n+1)/2$ conditions in number, whereas only n are necessary. Secondly, despite several ingenious methods devised by Routh, it is not easy to compute them in the general case.

Recognizing these difficulties, he therefore began anew from an entirely different angle. Utilizing a theorem of Cauchy concerning the relationship between the behavior of a polynomial on a closed contour in the complex domain and the number of roots within that closed curve, he derived necessary and sufficient conditions, which may be written in the slightly more convenient determinantal form of Hurwitz and Frazer and Duncan as follows:

$$T_0 = p_0 > 0, \qquad T_1 = p_1 > 0, \qquad T_2 = \begin{vmatrix} p_1 & p_3 \\ p_0 & p_2 \end{vmatrix} > 0,$$

$$(6) \qquad T_3 = \begin{vmatrix} p_1 & p_3 & p_5 \\ p_0 & p_2 & p_4 \\ 0 & p_1 & p_3 \end{vmatrix} > 0, \quad \cdots \quad T_s = \begin{vmatrix} p_1 & p_3 & \cdots & p_{2s-1} \\ p_0 & p_2 & \cdots & p_{2s-2} \\ 0 & p_1 & \cdots & p_{2s-3} \\ 0 & p_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p_s \end{vmatrix} > 0.$$

[1] E. J. Routh, *A Treatise on the Stability of a Given State of Motion*, (London, 1877), Chaps. 2 and 3; *Advanced Rigid Dynamics*, 6th ed., London, 1905, Chap. 6.

[2] Hurwitz, *Math. Ann.*, Vol. 46 (1895), p. 521.

[3] R. A. Frazer and W. J. Duncan, *Royal Soc. Proc.*, Series A, Vol. 124 (1929), p. 642. Also R. A. Frazer, W. J. Duncan, and A. R. Collar, *Elementary Matrices*, Cambridge University Press, 1938, pp. 151–155.

The law of formation of these determinants is obvious. In the first row the odd p's starting with the first are listed. Within each column the p's diminish one unit at a time. Any p with negative subscript derived by this formula is treated as zero, and all p's of subscript higher than the degree of the equation are set equal to zero. With this convention, for p_0 made positive, complete and independent necessary conditions are that all principal minors of T_n formed by deleting successively the last row and column must be positive. These conditions are n in number and are independent.

3. Complete, independent, necessary and sufficient conditions. Corresponding to Routh's first attack on the problem, we might consider an equation of degree $n(n-1)/2$ whose roots equal the *products* two at a time of the original equation's. If this equation and the original equation have *real* roots less than unity in absolute value, our problem is solved. This is guaranteed if, and only if, two further transformed equations with roots equal to the squares minus unity of the roots of the original and derived equations respectively all have positive coefficients. These conditions are necessary and sufficient, but not independent, and cannot be easily computed in the general case. Therefore, I follow Routh's example and approach the problem from a different point of view.

When the roots of $f(x) = 0$ are plotted in the complex plane, they must all lie within the unit circle if their absolute values are to be less than unity, and conversely. We might therefore attempt to apply Cauchy's theorem. However, it is not necessary to do so. Routh has shown what the conditions are that there be no roots in the right-hand half-plane. Can we find a complex transformation of variables which carries the unit circle into the left-hand half-plane?

The answer is in the affirmative. The linear complex transformation

$$x = \frac{z+1}{z-1}, \qquad z = \frac{x+1}{x-1} \tag{7}$$

will accomplish this. But after substituting for x its value in terms of z, we cease to have a polynomial but rather a rational function of z as follows:

$$f(x) = f\left(\frac{z+1}{z-1}\right) = \frac{\sum_{i=0}^{n} p_i (z+1)^{n-i}(z-1)^i}{(z-1)^n} = 0. \tag{8}$$

We need only consider the polynomial in the numerator, i.e.,

$$\varphi(z) = \sum_{0}^{n} \pi_i z^{n-i} = 0. \tag{9}$$

In order that the roots of the original equation be less than unity, in absolute value, it is necessary and sufficient that the real parts of the roots of equation (9) *be negative.* Once we determine the coefficients (π_i) in terms of the original p's, we can easily apply Routh's theorems. This yields $n + 1$ necessary and sufficient conditions, all of which are independent.

Expanding the numerator of the right-hand side of (8) and collecting terms, the following explicit formulas for the π's are directly obtained:

$$\pi_i = \sum_{j=0}^{n} p_j \sum_{k=0}^{m(i,j)} {}_{n-j}C_{i-k}(-1)^k {}_jC_k, \tag{10}$$

where

$${}_vC_w = \frac{v!}{(v-w)!\,w!},$$

and

$m(i, j) =$ the smaller of i and j.

For fourth and higher degree equations literal substitution, while always possible, results in complicated expressions. It is preferable, therefore, to compute the π's numerically and then apply the conditions of (6) directly.

Other necessary conditions can be easily derived, but they will be dependent upon these. Thus, each π must be positive; but this is not, by itself, sufficient. Or, adding π_0 and π_n we find

$$\pi_0 + \pi_n = p_0 + p_2 + p_4 + \cdots > 0, \tag{11}$$

i.e., the sum of the even p's must be positive. Similarly, still other linear sums of other π's will result in cancellation of certain of the p's. Except on special occasions there is probably no labor saved by utilizing conditions derived in this way.

One obvious but useful necessary condition will be stated without proof. If one forms polynomials from subsets of the coefficients of a given "stable" polynomial formed by arbitrary "cuts" which leave adjacent coefficients in unchanged order and introduce no gaps within each set, then the resulting polynomials will all be stable.

Special sufficiency conditions also can be developed. Carmichael[4] presents certain inequalities between the absolute values of the largest root and the coefficients of the original equation. For special problems these may be fruitfully applied.

4. Example. In conclusion I apply the conditions derived here to a well-known numerical equation determined statistically by Tinbergen[5] in the analysis of economic fluctuations. It is a fourth order difference equation with constant coefficients,

$$Z_t - .398Z_{t-1} + .220Z_{t-2} - .013Z_{t-3} - .027Z_{t-4} = 0 \tag{12}$$

[4] R. D. Carmichael, *Amer. Math. Soc. Bull.*, Vol. 24 (1918), pp. 286–296.

[5] J. Tinbergen, *Business Cycles in the United States, 1919–1932*, League of Nations, 1939, p. 140.

with the associated indicial equation

$$f(x) = x^4 - .398x^3 + .220x^2 - .013x - .027 = 0. \tag{13}$$

Its roots have been computed and are known to be less than unity in absolute value. This may be verified by computing

$$\begin{aligned} \pi_0 &= 0.782 > 0 \\ \pi_1 &= 3.338 > 0 \\ \pi_2 &= 5.398 > 0 \\ \pi_3 &= 4.878 > 0 \\ \pi_4 &= 1.604 > 0 \\ T_2 &= 14.204 > 0 \\ T_3 &= 43.177 > 0 \end{aligned} \tag{14}$$

To compute the same results by cross-multiplication the work is arranged as follows:

$$\begin{array}{ccc} \pi_0 & \pi_2 & \pi_4 \\ .782 & 5.398 & 1.604 \\ \pi_1 & \pi_3 & \\ 3.338 & 4.878 & \\ \pi_1\pi_2 - \pi_0\pi_3 & \pi_3\pi_4 - 0 & \\ 14.204 & 7.824 & \\ \pi_3(\pi_1\pi_2 - \pi_0\pi_3) - \pi_1\pi_3\pi_4 & & \\ 43.177 & & \end{array} \tag{15}$$

It may be remarked that the presence of a negative coefficient anywhere in the table is an immediate indication of instability, and that there is no necessity to continue the computation until a negative sign appears in a leading coefficient. This fact often saves much labor.

VALUES OF MILLS' RATIO OF AREA TO BOUNDING ORDINATE AND OF THE NORMAL PROBABILITY INTEGRAL FOR LARGE VALUES OF THE ARGUMENT

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A pair of simple inequalities is proved which constitute upper and lower bounds for the ratio R_x[1], valid for $x > 0$. The writer has failed to encounter these inequalities in the literature, hence it seems worthwhile to present them for whatever value they may have.

[1] J. P. Mills, "Table of ratio: area to bounding ordinate, for any portion of the normal curve." *Biometrika Vol. 18* (1926) pp. 395–400. Also Pearson's tables, Part II, Table III.

The function R_x is defined by

$$R_x = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt. \tag{1}$$

The following relations between $R = R_x$ and its derivatives are easily established by direct differentiations and substitutions:

$$\frac{dR}{dx} = xR - 1, \tag{2}$$

$$\frac{d^2R}{dx^2} = x\frac{dR}{dx} + R = \frac{x^2+1}{x}\frac{dR}{dx} + \frac{1}{x}, \tag{3}$$

$$\frac{d^3R}{dx^3} = \left(1 + \frac{2}{x^2+1}\right)x\frac{d^2R}{dx^2} - \frac{2}{x^2+1}. \tag{4}$$

Also by ordinary rules

$$R_x > 0, \tag{5}$$

$$\lim_{x\to\infty} xR_x = 1. \tag{6}$$

1°. Suppose that at any point $x_1 > 0$, $x_1R > 1$. Then by (2) $dR/dx > 0$, and R_x would continue to increase with increasing x: still more, xR_x would continue to increase, hence we should have $xR_x > 1$ for $x \geqq x_1$, which contradicts (6). Therefore we find $xR_x \leqq 1$ for $x > 0$, and

$$R_x \leqq \frac{1}{x}, \tag{7}$$

which establishes the required upper inequality.

2°. Suppose that at any point $x_2 > 0$, $d^2R/dx^2 < 0$. Then by (4) $d^3R/dx^3 = (d/dx)(d^2R/dx^2) < 0$ at this point. Since these derivatives are continuous this implies that for all $x > x_2$, $d^2R/dx^2 < [d^2R/dx^2]_{x=x_2} < 0$. Then we get the inequalities, for $x > x_2$

$$\frac{dR}{dx} < \left[\frac{dR}{dx}\right]_2 + (x - x_2)\left[\frac{d^2R}{dx^2}\right]_2 < \left[\frac{dR}{dx}\right]_2$$

$$R < R_{x_2} + (x - x_2)\left[\frac{dR}{dx}\right]_2 + \tfrac{1}{2}(x - x_2)^2\left[\frac{d^2R}{dx^2}\right]_2$$

where $[\ ]_2$ indicates evaluation at $x = x_2$. Since $[d^2R/dx^2]_2 < 0$, this implies that for sufficiently large x, $R_x < 0$, which contradicts (5). It follows then that (3) is positive, and substitution of (2) gives

$$R_x \geqq \frac{x}{x^2+1}. \tag{8}$$

We combine (7) and (8) in the double inequality:

$$\frac{x}{x^2+1} \leqq R_x \leqq \frac{1}{x}, \qquad \text{if } x \geqq 0. \tag{9}$$

This gives for the probability integral the corresponding inequality

$$\frac{x}{x^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leqq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \leqq \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{10}$$

It can easily be shown (for $x > 0$) that equalities in (9) and (10) are impossible.

≥ 0.

ible.